

Two-Dimensional Heisenberg Model with Nonlinear Interactions: $1/N$ Corrections

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Abstract

We investigate a two-dimensional classical N -vector model with a generic nearest-neighbor interaction $W(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$ in the large- N limit, focusing on the finite-temperature transition point at which energy-energy correlations become critical. We show that this transition belongs to the Ising universality class. However, the width of the region in which Ising behavior is observed scales as $1/N^{3/2}$ along the magnetic direction and as $1/N$ in the thermal direction; outside a crossover to mean-field behavior occurs. This explains why only mean-field behavior is observed for $N = \infty$.

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1 Introduction

The two-dimensional Heisenberg model has been the object of extensive studies which mainly focused on the $O(N)$ -symmetric Hamiltonian

$$H = -N\beta \sum_{\langle ij \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \quad (1)$$

where $\boldsymbol{\sigma}_i$ is an N -dimensional unit spin and the sum is extended over all lattice nearest neighbors. The behavior of this system in two dimensions is well understood. It is disordered for all finite β [1] and it is described for $\beta \rightarrow \infty$ by the perturbative renormalization group [2–4]. The square-lattice model has been extensively studied numerically [5–10], checking the perturbative predictions [11–14] and the nonperturbative constants [15–17].

Beside Hamiltonian (1), one can also consider more general interactions. In particular, one can consider the most general Hamiltonian with nearest-neighbor couplings given by

$$H = -N\beta \sum_{\langle ij \rangle} W(1 + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j), \quad (2)$$

where $W(x)$ is a generic function such that $W(2) > W(x)$ for all $0 \leq x < 2$, in order to guarantee that the system orders ferromagnetically for $\beta \rightarrow \infty$. The behavior of model (2) for $\beta \rightarrow \infty$ is similar to that of model (1): the critical behavior is described by the perturbative renormalization group, the spin-spin correlation length diverging as $\beta \rightarrow \infty$.

While for $\beta \rightarrow \infty$ the behavior is similar, for finite values of β the more general Hamiltonian (2) may give rise to a more complex phase diagram. In particular, one may observe first-order transitions, as it has been shown rigorously for a large class of functions $W(x)$ in Refs. [18, 19]. These rigorous results confirm the conclusions of Refs. [20–23] that found first-order transitions in the mixed $O(N)$ - RP^{N-1} for $N = \infty$, in a modified RP^{N-1} for $N = 3$, and in systems with $W(x) \sim x^p$ for p large enough for $N = 3$ and $N = \infty$ respectively. The presence of first-order transitions does not violate the Mermin-Wagner theorem [1], since the spin-spin correlation length is finite at the transition. However, the model studied in Ref. [20] for $N = \infty$ also shows a finite- β critical point. The specific heat diverges indicating that energy fluctuations become critical. On the other hand, the spin-spin correlation length is finite, although nonanalytic, at the transition, so that the Mermin-Wagner theorem [1] is not violated. For a long time it has not been clear whether the observed critical point was an artifact of the large- N limit or rather a true transition to be expected also for finite values of N [20, 24, 25]. Recently, Blöte, Guo, and Hilhorst [21] studied numerically a class of systems with $W(x) \sim x^p$ and $N = 3$ and found for a specific value of p a critical transition with essentially the same features as those observed in Ref. [20] for $N = \infty$. This confirms the existence of finite- β critical points where energy-energy correlations are critical, while the spin-spin correlation function is still short-ranged.

What remains to be understood is the nature of the critical point. A simple symmetry argument [21] indicates that the transition should belong to the Ising universality class for any value of N , and this is confirmed by the numerical results for $N = 3$. On the other hand, the large- N analysis [20, 22] finds mean-field exponents. In this paper we wish to reconcile these two apparently contradictory results. Indeed, we show that, as soon as large- N fluctuations are considered, the transition becomes an Ising one. The mechanism that

works in the large- N limit is very similar to that observed in medium-range models [26–30], with N playing the role of the interaction range. If $t \equiv (\beta - \beta_c(N))/\beta_c(N)$ is the reduced temperature and h is a properly defined scaling field that plays the role of the Ising magnetic field, we find that a universal scaling behavior is obtained by taking $t \rightarrow 0$, $h \rightarrow 0$, $N \rightarrow \infty$ and keeping fixed $\tilde{t} \equiv tN$, $\tilde{h} \equiv hN^{3/2}$. Ising behavior is observed only for $\tilde{t} \ll 1$ and $\tilde{h} \ll 1$, so that the width of the Ising critical region goes to zero as $N^{-3/2}$ (except in the thermal direction where it goes to zero as N^{-1}) and no Ising behavior is observed for $N = \infty$. In the other limit mean-field behavior is observed instead.

The analysis of the large- N corrections is quite complex. As in the standard $1/N$ expansion we first derive the effective Hamiltonian for the auxiliary fields. However, at the critical point the propagator is singular, forbidding a standard $1/N$ expansion. We single out the zero mode and show that the effective Hamiltonian that is obtained by integrating the massive modes corresponds to a weakly coupled ϕ^4 theory. The analysis is somewhat complicated by the fact that the \mathbb{Z}_2 symmetry is absent and thus there is no natural definition of the thermal and magnetic scaling fields. The correct scaling fields are linear combinations of the Hamiltonian parameters that must be computed order by order in $1/N$. A detailed field-theoretical analysis of the effective Hamiltonian allows us to compute the $1/N$ corrections to the position of the critical point and to express the large- N scaling behavior close to the transition in terms of Ising crossover functions.

The paper is organized as follows. In Sec. 2 we review the large- N limit for model (2) on a square lattice, reporting some basic formulas needed in the paper. Derivations are reported in Ref. [22]. In Sec. 3 we determine the linear scaling fields for $N = \infty$ and show that they are related by the mean-field equation of state. In Sec. 4 and 5 we determine the effective Hamiltonian for the auxiliary fields and for the zero mode. Sec. 6 discusses the weakly coupled ϕ^4 theory in the absence of \mathbb{Z}_2 symmetry, determining the relevant renormalization constants and the scaling behavior. Finally, in Sec. 7 we apply these results to the large- N limit. We determine the correct scaling fields, the $1/N$ corrections to the position of the critical point and the width of the Ising critical region. In Sec. 8 we explicitly give the scaling behavior for $\langle \sigma_x \sigma_{x+\mu} \rangle$, so that we can compare with numerical results, for instance with those of Ref. [21]. Finally, in Sec. 9 we report some numerical results for the mixed $O(N)$ - RP^{N-1} discussed in Ref. [20] and for the Hamiltonian introduced in Ref. [21]. Conclusions are presented in Sec. 10.

2 The large- N limit

We consider Hamiltonian (2) on a square lattice and normalize $W(x)$ by requiring $W'(2) = 1$ so that in the spin-wave limit

$$H = \frac{N\beta}{2} \int dx \partial_\mu \sigma \cdot \partial_\mu \sigma. \quad (3)$$

The standard N -vector model corresponds to $W(x) = x$. In this work we do not specify the function $W(x)$; we only assume that it depends on a parameter p and that in the (β, p) plane there is a first-order transition line $\beta = \beta_f(p)$ ending at a critical point (β_c, p_c) that will be characterized below.

Our discussion strictly applies only to ferromagnetic models, but under standard assumptions can be extended to RP^{N-1} models which correspond to functions satisfying $W(1+x) = W(1-x)$. In this case the Hamiltonian is invariant under the local transformations $\boldsymbol{\sigma}_x \rightarrow z_x \boldsymbol{\sigma}_x$, $z_x = \pm 1$.

The large- N limit has been discussed in detail in Ref. [22]. We introduce three auxiliary fields $\lambda_{x\mu}$, $\rho_{x\mu}$, and μ_x in order to linearize the dependence of the Hamiltonian on the spins $\boldsymbol{\sigma}$ and to eliminate the constraint $\boldsymbol{\sigma}_x^2 = 1$. The partition function becomes

$$Z = \int \prod_{x\mu} [d\rho_{x\mu} d\lambda_{x\mu}] \prod_x [d\mu_x d\boldsymbol{\sigma}_x] e^{-NA}, \quad (4)$$

where

$$A = -\frac{\beta}{2} \sum_{x\mu} [\lambda_{x\mu} + \lambda_{x\mu} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} - \lambda_{x\mu} \rho_{x\mu} + 2W(\rho_{x\mu})] + \frac{\beta}{2} \sum_x (\mu_x \boldsymbol{\sigma}_x^2 - \mu_x). \quad (5)$$

We perform a saddle-point integration by writing

$$\begin{aligned} \lambda_{x\mu} &= \alpha + \frac{1}{\sqrt{N}} \hat{\lambda}_{x\mu}, \\ \rho_{x\mu} &= \tau + \frac{1}{\sqrt{N}} \hat{\rho}_{x\mu}, \\ \mu_x &= \gamma + \frac{1}{\sqrt{N}} \hat{\mu}_x. \end{aligned} \quad (6)$$

In Ref. [22] we computed the corresponding saddle-point equations. They can be written as

$$\gamma = \alpha(4 + m_0^2)/2, \quad (7)$$

$$\alpha = 2W'(\tau), \quad (8)$$

$$\tau = \bar{\tau}(m_0^2) \equiv 2 + \frac{m_0^2}{4} - \frac{1}{4B_1(m_0^2)}, \quad (9)$$

$$\beta = \frac{B_1(m_0^2)}{W'(\bar{\tau}, p)}, \quad (10)$$

where the parameter m_0 is related to the spin-spin correlation length $\xi_\sigma = 1/m_0$ and

$$B_n(m_0^2) \equiv \int_{\mathbf{q}} \frac{1}{(\hat{q}^2 + m_0^2)^n}, \quad (11)$$

where the integral is extended over the first Brillouin zone.

In Ref. [22] it was shown that generic models may show first-order transitions. This happens when, for given β , there are several values of m_0^2 that solve the gap equation (10). Here, we will be interested at the endpoint of the first-order transition line $m_0 = m_{0c}$, $p = p_c$. At this point we have

$$\frac{\partial \beta}{\partial m_0^2} = 0, \quad \frac{\partial^2 \beta}{\partial (m_0^2)^2} = 0, \quad (12)$$

where the derivatives are taken at fixed p . We will only consider the generic case, always assuming that the third derivative is nonvanishing at the transition.

3 Equation of state and scaling fields

We wish now to parametrize the singular behavior for $\beta \rightarrow \beta_c$ and $p \rightarrow p_c$. Expanding the gap equation (10) near the critical point we obtain

$$\beta - \beta_c = \sum_{nm} a_{nm} (p - p_c)^n (m_0^2 - m_{0c}^2)^m. \quad (13)$$

Because of the definition of β_c we have $a_{00} = 0$. Moreover, Eq. (12) implies that $a_{01} = a_{02} = 0$. For $p = p_c$ we see that m_0^2 has the leading behavior

$$m_0^2 - m_{0c}^2 \approx \left(\frac{\beta - \beta_c}{a_{03}} \right)^{1/3}, \quad (14)$$

while for $\beta = \beta_c$, we have

$$m_0^2 - m_{0c}^2 \approx \left(-\frac{a_{10}(p - p_c)}{a_{03}} \right)^{1/3}. \quad (15)$$

The nonanalytic behavior with exponent $1/3$ is observed along any straight line approaching the critical point, except that satisfying $\beta - \beta_c - a_{10}(p - p_c) = 0$. Therefore, the correct linear scaling field is

$$u_h \equiv \beta - \beta_c - a_{10}(p - p_c), \quad (16)$$

and we have $m_0^2 - m_{0c}^2 \sim u_h^{1/3}$ whenever $u_h \neq 0$. To include the case $u_h = 0$, we write the general scaling equation

$$m_0^2 - m_{0c}^2 = u_h^{1/3} f(x), \quad (17)$$

where $f(x)$ is a scaling function and x is a scaling variable to be determined. Then, we use again the gap equation (13). Keeping only the leading terms we obtain

$$u_h = a_{20}(p - p_c)^2 + a_{11}(p - p_c)u_h^{1/3}f(x) + a_{03}u_h f(x)^3, \quad (18)$$

so that

$$a_{03}f(x)^3 + a_{11}(p - p_c)u_h^{-2/3}f(x) - a_{20}(p - p_c)^2u_h^{-1} - 1 = 0 \quad (19)$$

Since $(p - p_c)^2u_h^{-1} = [(p - p_c)u_h^{-2/3}]^2u_h^{1/3}$, the third term can be neglected. Thus, we may take (the prefactor has been introduced for later convenience)

$$x \equiv a_{11}(p - p_c)|u_h|^{-2/3}. \quad (20)$$

The function $f(x)$ satisfies

$$a_{03}f(x)^3 + xf(x) - 1 = 0. \quad (21)$$

Such an equation is exactly the mean-field equation for the magnetization. Indeed, if we consider the mean-field Hamiltonian

$$\mathcal{H} = -hM + \frac{t}{2}M^2 + \frac{u}{24}M^4, \quad (22)$$

the stationarity condition gives

$$-h + tM + \frac{u}{6}M^3 = 0, \quad (23)$$

which is solved by $M = h^{1/3}\hat{f}(t|h|^{-2/3})$, where $\hat{f}(x)$ satisfies Eq. (21) with $a_{03} = u/6$. It is thus clear that the scaling field (16) corresponds to h , while p corresponds to the temperature. Note that this identification is not unique, since only the line $H = 0$ is uniquely defined by the singular behavior. For instance, in the usual Ising case, we could define $t' = t + ah$ without changing the scaling equation of state, since $t|h|^{-2/3} = t'|h|^{-2/3} + ah^{1/3}$. Since the scaling limit is taken with $h \rightarrow 0$, $t \rightarrow 0$ at fixed $t|h|^{-2/3}$, we see that $t|h|^{-2/3} \approx t'|h|^{-2/3}$. In the Ising case, however, there is exact \mathbb{Z}_2 symmetry and thus the natural t variable is defined so that it is invariant under the symmetry. In our case we could define $u_t = p - p_c + A(\beta - \beta_c)$ and fix A by requiring the leading correction on any line (except $u_h = 0$) to be of order $u_h^{2/3}$ instead of order $u_h^{1/3}$, recovering in this way an approximate \mathbb{Z}_2 symmetry. For our purposes this is irrelevant and thus we will use $p - p_c$ as thermal scaling field.

Eq. (21) may have more than one solution. A simple analysis shows that, if $a_{03} > 0$, $f(x)$ is the unique positive solution of Eq. (21) and has the following asymptotic behaviors: $f(x) \approx \sqrt{-x/a_{03}}$ for $x \rightarrow -\infty$ and $f(x) \approx 1/x$ for $x \rightarrow \infty$. If a_{03} is negative the same results apply to $-f(-x)$.

Eq. (17) gives the leading behavior. It is also possible to compute the subleading corrections. We obtain for the leading one

$$m_0^2 - m_{0c}^2 = u_h^{1/3}f(x) + u_h^{2/3}g(x) + O(u_h), \quad (24)$$

with

$$g(x) = -\frac{a_{20}x^2 + a_{12}a_{11}xf(x)^2 + a_{04}a_{11}^2f(x)^4}{a_{11}^2(x + 3a_{03}f(x)^2)}. \quad (25)$$

Finally, let us discuss the singular behavior of the energy. We have

$$E = 2W(\bar{\tau}, p). \quad (26)$$

Such a function is regular in m_0^2 and p . Since $p - p_c \sim x|u_h|^{2/3}$ and $m_0^2 - m_{0c}^2 \sim u_h^{1/3}$, the leading term is obtained by expanding the previous equation in powers of $m_0^2 - m_{0c}^2$. Thus

$$E = 2W_c + 2W'_c \left. \frac{B_1^2 - B_2}{4B_1^2} \right|_{m_0=m_{0c}} u_h^{1/3}f(x) + O(u_h^{2/3}). \quad (27)$$

where $W' = \partial W(x, p)/\partial x$ with $x = \bar{\tau}$ and the suffix c indicates that all quantities must be computed at the critical point.

4 The $1/N$ calculation: propagator and effective vertices

In order to perform the $1/N$ calculation we integrate out the fields σ , which is straightforward since the Hamiltonian is quadratic in these fields. We thus obtain

$$Z = \int \prod_{x\mu} [d\rho_{x\mu} d\lambda_{x\mu}] \prod_x d\mu_x e^{-\mathcal{H}}. \quad (28)$$

where the effective Hamiltonian \mathcal{H} can be expanded in powers of the auxiliary fields. If we define a five-component field Ψ_A

$$\Psi = (\hat{\mu}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\rho}_1, \hat{\rho}_2), \quad (29)$$

then \mathcal{H} can be written as

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \int_{\mathbf{p}} \sum_{A_1 A_2} \Psi_{A_1}(-\mathbf{p}) P_{A_1 A_2}^{-1}(\mathbf{p}) \Psi_{A_2}(\mathbf{p}) \\ & + \sum_{n=3} \frac{1}{n!} \frac{1}{N^{n/2-1}} \int_{\mathbf{p}_1} \cdots \int_{\mathbf{p}_n} \delta \left(\sum_i \mathbf{p}_i \right) \sum_{A_1, \dots, A_n} V_{A_1, \dots, A_n}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) \Psi_{A_1}(\mathbf{p}_1) \cdots \Psi_{A_n}(\mathbf{p}_n), \end{aligned} \quad (30)$$

where the indices A_i run from 1 to 5,

$$\int_{\mathbf{p}} \equiv \int \frac{d^2 p}{(2\pi)^2}, \quad \delta(\mathbf{p}) \equiv \prod_{\alpha=1}^2 2\pi \delta(p_i), \quad (31)$$

and the integration is over the first Brillouin zone. The propagator can be explicitly written as¹

$$\mathbf{P}^{-1}(\mathbf{p}) = -\frac{1}{W'^2} \begin{pmatrix} \frac{1}{2}A_{0,0} & -\frac{1}{2}A_{1,0} & -\frac{1}{2}A_{0,1} & 0 & 0 \\ -\frac{1}{2}A_{1,0} & \frac{1}{2}A_{2,0} & \frac{1}{2}A_{1,1} & -\frac{1}{2}\beta W'^2 & 0 \\ -\frac{1}{2}A_{0,1} & \frac{1}{2}A_{1,1} & \frac{1}{2}A_{0,2} & 0 & -\frac{1}{2}\beta W'^2 \\ 0 & -\frac{1}{2}\beta W'^2 & 0 & \beta W''W'^2 & 0 \\ 0 & 0 & -\frac{1}{2}\beta W'^2 & 0 & \beta W''W'^2 \end{pmatrix}, \quad (32)$$

where W should always be intended as a function of $\bar{\tau}(m_0^2)$,

$$A_{i,j}(\mathbf{p}, m_0^2) \equiv \int_{\mathbf{q}} \frac{\cos^i q_x \cos^j q_y}{[m_0^2 + \widehat{(q + \frac{p}{2})}]^2 [m_0^2 + \widehat{(q - \frac{p}{2})}]^2}, \quad (33)$$

and $\hat{p}^2 \equiv 4(\sin^2 p_x/2 + \sin^2 p_y/2)$.

For $p \rightarrow 0$, by using the algebraic algorithm described in App. A of Ref. [13], it is easy to express the integrals $A_{i,j}(0, m_0^2)$ in terms of the integrals $B_n(m_0^2)$ defined in Eq. (11) with $n = 1, 2$. Explicitly we have

$$\begin{aligned} A_{00}(\mathbf{0}, m_0^2) &= B_2, \\ A_{10}(\mathbf{0}, m_0^2) &= A_{01}(\mathbf{0}, m_0^2) = \left(1 + \frac{m_0^2}{4}\right) B_2 - \frac{1}{4} B_1, \\ A_{11}(\mathbf{0}, m_0^2) &= -\frac{1}{8}(4 + m_0^2) B_1 + \frac{1}{8}(8 + 8m_0^2 + m_0^4) B_2, \\ A_{20}(\mathbf{0}, m_0^2) &= A_{02}(\mathbf{0}, m_0^2) = \frac{1}{8} + B_2 - \frac{1}{8}(4 + m_0^2) B_1. \end{aligned} \quad (34)$$

¹It is useful to write the Fourier transform of $\hat{\lambda}_x$ as $\hat{\lambda}(\mathbf{p}) = e^{-ip_\mu/2} \sum_x e^{-ip \cdot x} \hat{\lambda}_x$. This makes all vertices and propagators real.

Vertices are analogously computed. It is easy to check that the only nonvanishing contributions for which some A_i is equal to 4 or 5 are

$$V_{4\dots 4}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) = V_{5\dots 5}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) = -\beta W^{(n)}(\bar{\tau}). \quad (35)$$

If all indices satisfy $A_i \leq 3$, then

$$\begin{aligned} & V_{A_1, \dots, A_n}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) \delta\left(\sum_i \mathbf{p}_i\right) = \\ &= \frac{(-1)^{n+1}}{[W'(\bar{\tau})]^n} \left\{ \prod_{i=1}^n \left[\int_{\mathbf{q}_i} \delta(\mathbf{q}_{i+1} - \mathbf{q}_i - \mathbf{p}_i) \frac{1}{q_i^2 + m_0^2} R_{A_i}(\mathbf{p}_i, \mathbf{q}_i) \right] + \text{permutations} \right\}, \end{aligned} \quad (36)$$

where

$$R_1(\mathbf{p}, \mathbf{q}) = 1, \quad R_2(\mathbf{p}, \mathbf{q}) = -\cos(q_x + p_x/2), \quad R_3(\mathbf{p}, \mathbf{q}) = -\cos(q_y + p_y/2). \quad (37)$$

The permutations should made the quantity in braces symmetric under any exchange of (\mathbf{p}_i, A_i) (the total number of needed terms is $(n-1)!/2$). As already discussed in Ref. [22], at the critical point the inverse propagator at zero momentum has a vanishing eigenvalue. Indeed, a straightforward computation gives

$$\det \mathbf{P}^{-1}(\mathbf{0}) = K_{0,\det} s_1, \quad (38)$$

where $K_{0,\det}$ is given by

$$K_{0,\det} \equiv -\frac{B_1^3}{128W'^6} [4B_1W' - (1 - m_0^2(8 + m_0^2)B_2)W''], \quad (39)$$

and

$$s_1 \equiv -\frac{\partial \beta}{\partial m_0^2} = \frac{1}{4B_1W'^2} (4B_1B_2W' + B_1^2W'' - B_2W''). \quad (40)$$

Eq. (38) shows that the determinant vanishes at the critical point—hence there is at least one vanishing eigenvalue—since there $\partial \beta / \partial m_0^2 = 0$. The corresponding eigenvector can be written as

$$\mathbf{z} = \left(2 \frac{A_{01}(\mathbf{0}, m_0^2)}{A_{00}(\mathbf{0}, m_0^2)}, 1, 1, \frac{1}{2W''}, \frac{1}{2W''} \right) \quad (41)$$

computed at the critical point. Indeed,

$$\sum_{B=1}^5 (P^{-1})_{AB}(\mathbf{0}) z_B = \frac{B_1}{4B_2W''} s_1 (0, 1, 1, 0, 0)_A. \quad (42)$$

Note that there is always only one zero mode. Indeed, at the critical point, we have from Eq. (40)

$$W'' = -\frac{4B_1B_2}{B_1^2 - B_2} W', \quad (43)$$

so that we can write at criticality

$$K_{0,\det} = -\frac{B_1^4 [B_1^2 - (8 + m_0^2)m_0^2 B_2^2]}{32(B_1^2 - B_2)} \frac{1}{W'^5}. \quad (44)$$

We have verified numerically that the prefactor of $1/W'^5$ is always finite and negative, so that $K_{0,\det}$ is always nonvanishing. Thus, for $p = p_c$ the determinant $\det \mathbf{P}^{-1}(\mathbf{0})$ vanishes as $(m_0^2 - m_0^2)^2$, as the eigenvalue associated with the zero mode, cf. Eq. (42). Thus, there can only be a single eigenvector with zero eigenvalue.

Because of the zero mode, it is natural to express the fields in terms of a new basis. For each m_0^2 and p , given the inverse propagator $P_{AB}^{-1}(\mathbf{p})$, there exists an orthogonal matrix $U(\mathbf{p}; m_0^2, p)$ such that $U^T P^{-1} U$ is diagonal. If $v_A(\mathbf{p}; m_0^2, p) \equiv U_{A1}(\mathbf{p}; m_0^2, p)$ is the eigenvector that correspond to the zero eigenvalue for $\mathbf{p} = 0$ at the critical point and $Q_{Aa}(\mathbf{p}; m_0^2, p) \equiv U_{A,a+1}(\mathbf{p}; m_0^2, p)$, $a = 1, \dots, 4$ are the other eigenvectors, we define new fields $\Phi_A(\mathbf{p})$ by writing

$$\Psi_A(\mathbf{p}) \equiv \sum_B U_{AB}(\mathbf{p}) \Phi_B(\mathbf{p}) = v_A(\mathbf{p}) \phi(\mathbf{p}) + \sum_a Q_{Aa}(\mathbf{p}) \varphi_a(\mathbf{p}), \quad (45)$$

where $\Phi = (\phi, \varphi_a)$. Eq. (45) defines the fields Φ_A up to a sign. For definiteness we shall assume $v_A(\mathbf{p})$ to be such that, at the critical point,

$$v_A(\mathbf{0}) = z_A / (\sum_B z_B^2)^{1/2}. \quad (46)$$

We do not specify the sign of φ_a since it will not play any role in the following.

The new field ϕ corresponds to the zero mode, while the four fields φ_a are the noncritical (massive) modes. The effective Hamiltonian for the fields Φ has an expansion analogous to that presented in Eq. (30) for Ψ . The propagator $\hat{P}_{AB}(\mathbf{p})$ of Φ is

$$\hat{P}_{AB}(\mathbf{p}) = \sum_{CD} P_{CD}(\mathbf{p}) U_{CA}(\mathbf{p}) U_{DB}(\mathbf{p}), \quad (47)$$

while the effective vertices are related to the previous ones by

$$\hat{V}_{A_i, \dots, A_n}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) = \sum_{B_1, \dots, B_n} V_{B_1, \dots, B_n}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) U_{B_1 A_1}(\mathbf{p}_1; m_0, p) \cdots U_{B_n A_n}(\mathbf{p}_n; m_0, p). \quad (48)$$

Note that $\hat{P}_{AB}(\mathbf{p})$ is diagonal by definition, i.e., $\hat{P}_{AB}(\mathbf{p}) = \delta_{AB} \hat{P}_{AA}(\mathbf{p})$. Relation (38) implies

$$\hat{P}_{11}(\mathbf{0}) \sim s_1 \sim (p - p_c), (m_0^2 - m_{0c}^2)^2 \quad (49)$$

close to the critical point. In Appendix A we prove a very general set of identities among the effective vertices. In particular, we show that at the critical point we also have

$$\hat{V}_{111}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0. \quad (50)$$

This result will play an important role in the following.

5 Effective Hamiltonian for the zero mode

As we have discussed in the previous Section, the propagator P has a zero mode at the critical point. Thus, in a neighbourhood of the critical point the standard $1/N$ expansion breaks down and a much more careful treatment of the zero mode is needed. For this purpose, we

are now going to integrate out the massive modes, obtaining an effective Hamiltonian for the field ϕ . More precisely, we define

$$e^{-\mathcal{H}_{\text{eff}}[\phi]} = \int \prod_{xa} d\varphi_{xa} e^{-\mathcal{H}[\Phi]}. \quad (51)$$

The effective Hamiltonian $\mathcal{H}_{\text{eff}}[\phi]$ has an expansion of the form

$$\begin{aligned} \mathcal{H}_{\text{eff}}[\phi] &= \frac{1}{\sqrt{N}} \tilde{H} \phi(\mathbf{0}) + \frac{1}{2} \int_{\mathbf{p}} \phi(-\mathbf{p}) \tilde{P}^{-1}(\mathbf{p}) \phi(\mathbf{p}) \\ &+ \sum_{n=3} \frac{1}{n!} \frac{1}{N^{n/2-1}} \int_{\mathbf{p}_1} \cdots \int_{\mathbf{p}_n} \delta\left(\sum_i \mathbf{p}_i\right) \tilde{V}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) \phi(\mathbf{p}_1) \cdots \phi(\mathbf{p}_n), \end{aligned} \quad (52)$$

where vertices and propagators also depend on N and have an expansion of the form

$$\begin{aligned} \tilde{H} &= \sum_{m=0} \frac{1}{N^m} \tilde{H}_m, \\ \tilde{P}^{-1}(\mathbf{p}) &= \sum_{m=0} \frac{1}{N^m} \tilde{P}_m^{-1}(\mathbf{p}), \\ \tilde{V}^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n) &= \sum_{m=0} \frac{1}{N^m} \tilde{V}_m^{(n)}(\mathbf{p}_1, \dots, \mathbf{p}_n). \end{aligned} \quad (53)$$

We report here the explicit expressions that we shall need in the following:

$$\tilde{H}_0 = \frac{1}{2} \int_{\mathbf{p}} \sum_{ab} \hat{V}_{1ab}(\mathbf{0}, \mathbf{p}, -\mathbf{p}) \hat{P}_{ab}(\mathbf{p}), \quad (54)$$

$$\tilde{P}_0^{-1}(\mathbf{p}) = \hat{P}_{11}^{-1}(\mathbf{p}), \quad (55)$$

$$\begin{aligned} \tilde{P}_1^{-1}(\mathbf{p}) &= \frac{1}{2} \int_{\mathbf{q}} \left[\sum_{ab} \hat{V}_{11ab}^{(4)}(\mathbf{p}, -\mathbf{p}, \mathbf{q}, -\mathbf{q}) \hat{P}_{ab}(\mathbf{q}) - \right. \\ &\quad \sum_{abcd} \hat{V}_{11a}^{(3)}(\mathbf{p}, -\mathbf{p}, \mathbf{0}) \hat{P}_{ab}(\mathbf{0}) \hat{V}_{bcd}^{(3)}(\mathbf{0}, \mathbf{q}, -\mathbf{q}) \hat{P}_{cd}(\mathbf{q}) - \\ &\quad \left. \sum_{abcd} \hat{V}_{1ab}^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \hat{V}_{1cd}^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \hat{P}_{ac}(\mathbf{q}) \hat{P}_{bd}(\mathbf{p} + \mathbf{q}) \right], \end{aligned} \quad (56)$$

$$\tilde{V}_0^{(3)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \hat{V}_{111}^{(3)}(\mathbf{p}, \mathbf{q}, \mathbf{r}), \quad (57)$$

$$\begin{aligned} \tilde{V}_0^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) &= \hat{V}_{1111}^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) - \\ &\quad \sum_{ab} \hat{V}_{11a}^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \hat{V}_{11b}^{(3)}(\mathbf{r}, \mathbf{s}, -\mathbf{r} - \mathbf{s}) \hat{P}_{ab}(\mathbf{p} + \mathbf{q}) + \text{two permutations}, \end{aligned} \quad (58)$$

$$\begin{aligned} \tilde{V}_0^{(5)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}) &= \hat{V}_{11111}^{(5)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}) - \\ &\quad \left[\sum_{ab} \hat{V}_{11a}^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \hat{V}_{111b}^{(4)}(\mathbf{r}, \mathbf{s}, \mathbf{t}, -\mathbf{r} - \mathbf{s} - \mathbf{t}) \hat{P}_{ab}(\mathbf{p} + \mathbf{q}) + 9 \text{ permutations} \right] + \\ &\quad \left[\sum_{abcd} \hat{V}_{11a}^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \hat{V}_{1bc}^{(3)}(\mathbf{r}, \mathbf{p} + \mathbf{q}, \mathbf{s} + \mathbf{t}) \hat{V}_{11d}^{(3)}(\mathbf{s}, \mathbf{t}, -\mathbf{s} - \mathbf{t}) \times \right. \\ &\quad \left. \times \hat{P}_{ab}(\mathbf{p} + \mathbf{q}) \hat{P}_{cd}(\mathbf{s} + \mathbf{t}) + 14 \text{ permutations} \right], \end{aligned} \quad (59)$$

where a, b, c, d run from 1 to 4 over the massive modes.

The identities presented in App. A allow us to derive several relations among the effective vertices at the critical point. We have for $p = p_c$ and $m_0^2 = m_{0c}^2$

$$\tilde{P}_0^{-1}(\mathbf{0}) = \frac{\partial \tilde{P}_0^{-1}(\mathbf{0})}{\partial m_0^2} = 0, \quad (60)$$

$$\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0, \quad (61)$$

$$\left[\frac{\partial}{\partial m_0^2} \tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) \right]^2 = \frac{\partial^2 \tilde{P}_0^{-1}(\mathbf{0})}{\partial (m_0^2)^2} \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}). \quad (62)$$

Relation (60) clearly shows that the standard $1/N$ expansion fails close to the critical point. Indeed, outside the critical point, $\tilde{P}_0^{-1}(\mathbf{p})$ is nonsingular and for large N it is enough to expand the interaction Hamiltonian in powers of $1/N$. On the other hand, this is not possible at the critical point. Since also the three-leg vertex vanishes at zero momentum in this case, cf. Eq. (61), the zero-momentum leading term is the quartic one. Since the coupling constant is proportional to $1/N$, the model effectively corresponds to a weakly coupled ϕ^4 theory. In order to have a stable ϕ^4 theory, we must also have $\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) > 0$. For a generic solution of the gap equations satisfying Eq. (12), this is not *a priori* guaranteed. Note, however, that if $\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) < 0$, then, for $p = p_c$, we have $\tilde{P}_0^{-1}(\mathbf{0}) \approx a(m_0^2 - m_{0c}^2)^2$, with $a < 0$, as a consequence of Eq. (62): the propagator has a *negative* mass for $N = \infty$. We believe—but we have not been able to prove—that such a phenomenon signals the fact that the solution we are considering is not the relevant one. We expect the existence of another solution of the gap equation (10) with a lower free energy.

The weakly coupled ϕ^4 theory shows an interesting crossover limit. If one neglects fluctuations it corresponds to tune p and m_0^2 so that \tilde{H} and $\tilde{P}_0^{-1}(\mathbf{0})$ go to zero as $N \rightarrow \infty$, in such a way that $\tilde{H}N$ and $\tilde{P}_0^{-1}(\mathbf{0})N$ remain constant. In this limit, Ising behavior is observed when the two scaling variables go to zero, while mean-field behavior is observed in the opposite case. Fluctuations change the simple scaling forms reported above and one must consider two additive renormalizations. A complete discussion will be reported below in Sec. 6.

Finally, we wish to change again the definition of the field so that the effective zero-momentum three-leg vertex vanishes for all p and m_0^2 in the limit $N \rightarrow \infty$. For this purpose we now define a new field

$$\alpha\chi(\mathbf{p}) = \phi(\mathbf{p}) + k\delta(\mathbf{p}), \quad (63)$$

where α and k are functions of p and m_0^2 to be fixed. The function k is fixed by requiring that the large- N zero-momentum three-leg vertex vanishes. Apparently, all $\tilde{V}^{(n)}$ contribute in this calculation. However, because of Eq. (61), $\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})$ vanishes at the critical point. As we already mentioned the interesting limit corresponds to considering $\Delta_m \equiv m_0^2 - m_{0c}^2 \rightarrow 0$ and $\Delta_p \equiv p - p_c \rightarrow 0$ together with $N \rightarrow \infty$. We will show in Sec. 7 that this limit should be taken keeping fixed $\Delta_m N^{1/3}$ and $\Delta_p N$, so that $\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})$ is effectively of order $N^{-1/3}$. Therefore, the equation defining k can be written in a compact form as

$$\frac{a_3}{N^{5/6}} + \sum_{n \geq 4} \frac{a_n k^{n-3}}{N^{n/2-1}} = 0, \quad (64)$$

where $a_n \approx a_{n0} + a_{n1}/N$ is the contribution of the n -point vertex at zero momentum. Eq. (64) can be rewritten as

$$\sum_{n=1}^{\infty} \frac{a_{n+3}}{N^{(n-1)/3}} \left(\frac{k}{N^{1/6}} \right)^n = -a_3 \quad (65)$$

which shows that k has an expansion of the form

$$k = k_0 N^{1/6} [1 + k_1 N^{-1/3} + O(N^{-2/3})]. \quad (66)$$

The leading constant k_0 depends only on the three- and four-leg vertex, the constant k_1 depends also on the five-leg vertex, and so on. The explicit calculation gives

$$\begin{aligned} k = & \sqrt{N} \frac{\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})}{\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})} + \\ & + \frac{\sqrt{N}}{2} \frac{\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})^2 \tilde{V}_0^{(5)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})}{[\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})]^3} + O(N^{-1/2}). \end{aligned} \quad (67)$$

By performing this rescaling, the k -leg ϕ -vertex that scales with N as $N^{1-k/2}$ (except for $k \leq 3$) gives a contribution to the m -leg χ vertex (of course $m \leq k$) of order $N^{1-m/2+(m-k)/3}$, which is therefore always subleading. Taking this result into account we obtain

$$\begin{aligned} \mathcal{H}_{\text{eff}} = & H \chi(\mathbf{0}) + \frac{1}{2} \int_{\mathbf{p}} \chi(\mathbf{p}) \chi(-\mathbf{p}) \bar{P}^{-1}(\mathbf{p}) + \\ & + \frac{1}{3! \sqrt{N}} \int_{\mathbf{p}, \mathbf{q}} \bar{V}^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \chi(\mathbf{p}) \chi(\mathbf{q}) \chi(-\mathbf{p} - \mathbf{q}) \\ & + \frac{1}{4! N} \int_{\mathbf{p}, \mathbf{q}, \mathbf{r}} \bar{V}^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, -\mathbf{p} - \mathbf{q} - \mathbf{r}) \chi(\mathbf{p}) \chi(\mathbf{q}) \chi(\mathbf{r}) \chi(-\mathbf{p} - \mathbf{q} - \mathbf{r}) + \dots \end{aligned} \quad (68)$$

where

$$\begin{aligned} H = & \frac{\alpha \tilde{H}_0}{\sqrt{N}} - \alpha k \left[\tilde{P}_0^{-1}(\mathbf{0}) + \frac{1}{N} \tilde{P}_1^{-1}(\mathbf{0}) \right] + \frac{\alpha k^2}{2 \sqrt{N}} \tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) - \frac{\alpha k^3}{6N} \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \\ & + \frac{\alpha k^4}{24 N^{3/2}} \tilde{V}_0^{(5)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) + O(N^{-7/6}), \end{aligned}$$

$$\begin{aligned} \bar{P}^{-1}(\mathbf{p}) = & \alpha^2 \tilde{P}_0^{-1}(\mathbf{p}) + \frac{\alpha^2}{N} \tilde{P}_1^{-1}(\mathbf{p}) - \frac{\alpha^2 k}{\sqrt{N}} \tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p}) + \frac{\alpha^2 k^2}{2N} \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p}) \\ & - \frac{\alpha^2 k^3}{6 N^{3/2}} \tilde{V}_0^{(5)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p}) + O(N^{-4/3}), \end{aligned}$$

$$\bar{V}^{(3)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \alpha^3 \tilde{V}_0^{(3)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) - \frac{\alpha^3 k}{\sqrt{N}} \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{p}, \mathbf{q}, \mathbf{r}) + O(N^{-2/3}),$$

$$\bar{V}^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) = \alpha^4 \tilde{V}_0^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) + O(N^{-1/3}), \quad (69)$$

where $\alpha = \alpha_0 + \alpha_1 N^{-2/3} + O(N^{-4/3})$ is fixed by requiring that

$$\bar{P}^{-1}(\mathbf{p}) - \bar{P}^{-1}(\mathbf{0}) \equiv K(\mathbf{p}) \approx \mathbf{p}^2, \quad (70)$$

for $\mathbf{p} \rightarrow 0$.

6 Intermezzo: The critical crossover limit

In this section we present a general discussion of the critical behavior of a generic weakly coupled lattice ϕ^4 theory, as is the effective Hamiltonian (68) for the zero mode. In particular, we show the existence of a critical limit, the critical crossover limit, that allows one to interpolate between the Ising behavior and the mean-field behavior. The arguments presented here closely follow the discussion of Ref. [29] for medium-range models. The main difference is the appearance of odd ϕ^3 interactions that require additional renormalizations.

6.1 General considerations

We wish now to discuss the critical behavior of a model with Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{eff}} = & H\varphi(\mathbf{0}) + \frac{1}{2} \int_{\mathbf{p}} [K(\mathbf{p}) + r] \varphi(\mathbf{p}) \varphi(-\mathbf{p}) \\ & + \frac{\sqrt{u}}{3!} \int_{\mathbf{p}} \int_{\mathbf{q}} V^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q}) \varphi(\mathbf{p}) \varphi(\mathbf{q}) \varphi(-\mathbf{p} - \mathbf{q}) \\ & + \frac{u}{4!} \int_{\mathbf{p}} \int_{\mathbf{q}} \int_{\mathbf{s}} V^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{s}, -\mathbf{p} - \mathbf{q} - \mathbf{s}) \varphi(\mathbf{p}) \varphi(\mathbf{q}) \varphi(\mathbf{s}) \varphi(-\mathbf{p} - \mathbf{q} - \mathbf{s}) \end{aligned} \quad (71)$$

on a square lattice. Here the integrations are extended over the first Brillouin zone, $\varphi(\mathbf{p})$ is the Fourier transform of the fundamental field which is normalized so that $K(\mathbf{p}) \approx \mathbf{p}^2$ for $\mathbf{p} \rightarrow 0$. By properly normalizing u (which is assumed to be positive) we can also require $V^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = 1$. We shall only consider the case in which the three-leg vertex vanishes at zero external momenta, i.e. $V^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0$. This is an important simplifying assumption. Indeed, this means that the cubic interaction has the form $\varphi^2 \square \varphi$, which is a renormalization-group irrelevant term according to power counting.

If $V^{(3)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0$ for all momenta, there is an interesting critical limit, the so-called critical crossover limit [31]. Indeed, one can show that, by properly defining a function $r_c(u)$, in the limit $u \rightarrow 0$, $t \equiv r - r_c(u) \rightarrow 0$, and $H \rightarrow 0$ at fixed H/u and t/u , the n -point susceptibility χ_n , i.e. the zero-momentum n -point connected correlation function, has the scaling form

$$\chi_n \approx ut^{-n} f_n^{\text{symm}}(t/u, H/u). \quad (72)$$

The scaling function $f_n^{\text{symm}}(x, y)$ is universal, i.e. it does not depend on the explicit form of $K(\mathbf{p})$ and of $V^{(4)}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ as long as the normalization conditions fixed above are satisfied. The definition of $r_c(u)$ has been discussed in Ref. [31] for the continuum model and in Ref. [29] for the more complex case of medium-range models. It is of order u and is obtained by requiring χ_2 to scale according to Eq. (72) in the critical crossover limit. If $r_c(u) = r_1 u + O(u^2)$, then at one loop we have

$$\chi_2 = \frac{1}{t} - \frac{u}{t^2} \left[r_1 + \frac{1}{2} \int_{\mathbf{p}} \frac{V^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p})}{K(\mathbf{p}) + t} \right] + O(u^2). \quad (73)$$

For $t \rightarrow 0$

$$\int_{\mathbf{p}} \frac{V^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p})}{K(\mathbf{p}) + t} \approx \int_{\mathbf{p}} \frac{1}{\hat{p}^2 + t} + \text{constant} = -\frac{1}{4\pi} \ln t + \text{constant}, \quad (74)$$

where $\hat{p}^2 = 4 \sin^2(p_1/2) + 4 \sin^2(p_2/2)$. Thus, we obtain

$$\chi_2 = \frac{1}{t} - \frac{u}{t^2} \left[r_1 - \frac{1}{8\pi} \ln u - \frac{1}{8\pi} \ln \frac{t}{u} + \text{constant} \right] + O(u^2). \quad (75)$$

Therefore, if we define

$$r_c(u) = \frac{u}{8\pi} \ln u, \quad (76)$$

the susceptibility χ_2 scales according to Eq. (72). No other correction is needed when considering the higher-order contributions and thus Eq. (76) gives an exact nonperturbative definition of $r_c(u)$. Such a result can be easily understood. In the continuum two-dimensional theory there is only one primitively divergent graph, the one-loop tadpole, and therefore only a one-loop mass counterterm is needed to make the theory finite. The same holds for the lattice model in the critical crossover limit, which is the limit in which the lattice theory goes over to the continuum model.

Now, let us consider the contributions of the three-leg vertex. Since this is a renormalization-group irrelevant term, the only changes we expect concern the renormalization of the bare parameters. Indeed, we will now show that, if one properly defines functions $r_c(u)$ and $h_c(u)$, then in the limit $u \rightarrow 0$, $t \equiv r - r_c(u) \rightarrow 0$, and $h \equiv H - h_c(u) \rightarrow 0$ at fixed h/u and t/u , the n -point susceptibility χ_n has the scaling form (72) with h replacing H and with the same scaling functions of the symmetric case.

We will first prove this result at two loops and then we will give a general argument that applies to all perturbative orders. Notice that it is enough to consider the case $h = 0$. Indeed, if Eq. (72) is valid for $h = 0$ and any n , then

$$\begin{aligned} \chi_n(h) &= \sum_{m=0} \frac{1}{m!} \chi_{n+m}(h=0) h^m \approx \sum_{m=0} \frac{1}{m!} u t^{-n-m} f_{n+m}^{\text{symm}}(t/u, 0) h^m = \\ &= u t^{-n} \sum_{m=0} \frac{1}{m!} f_{n+m}^{\text{symm}}(t/u, 0) (t/u)^{-m} (h/u)^m. \end{aligned} \quad (77)$$

which proves Eq. (72) for all values of h .

6.2 Explicit two-loop perturbative calculation

At tree level, all contributions to χ_n that contain three-leg vertices vanish because $V^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) = 0$. Therefore, χ_n is identical to χ_n in the \mathbb{Z}_2 -symmetric φ^4 theory; in particular $\chi_{2n+1} = 0$. At one-loop order, graphs contributing to χ_n are formed by a single loop made of a three-leg vertices and of b four-leg vertices, with $a + 2b = n$. Each of them contributes a term of the form

$$\frac{u^{n/2}}{t^n} \int_{\mathbf{p}} \frac{V_3^a(\mathbf{p}) V_4^b(\mathbf{p})}{[K(\mathbf{p}) + t]^{a+b}}, \quad (78)$$

where $V_3(\mathbf{p}) \equiv V^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p})$ and $V_4(\mathbf{p}) \equiv V^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p})$. The leading contribution for $t \rightarrow 0$ is obtained by replacing each quantity with its small- \mathbf{p} behavior, i.e. $V_4(\mathbf{p}) \approx 1$, $V_3(\mathbf{p}) \sim \hat{p}^2$, and $K(\mathbf{p}) \approx \hat{p}^2$, with $\hat{p}^2 = 4 \sin^2(p_1/2) + 4 \sin^2(p_2/2)$. Then, we obtain a contribution proportional to

$$\frac{u^{n/2}}{t^n} \int_{\mathbf{p}} \frac{(\hat{p}^2)^a}{(\hat{p}^2 + t)^{a+b}}. \quad (79)$$

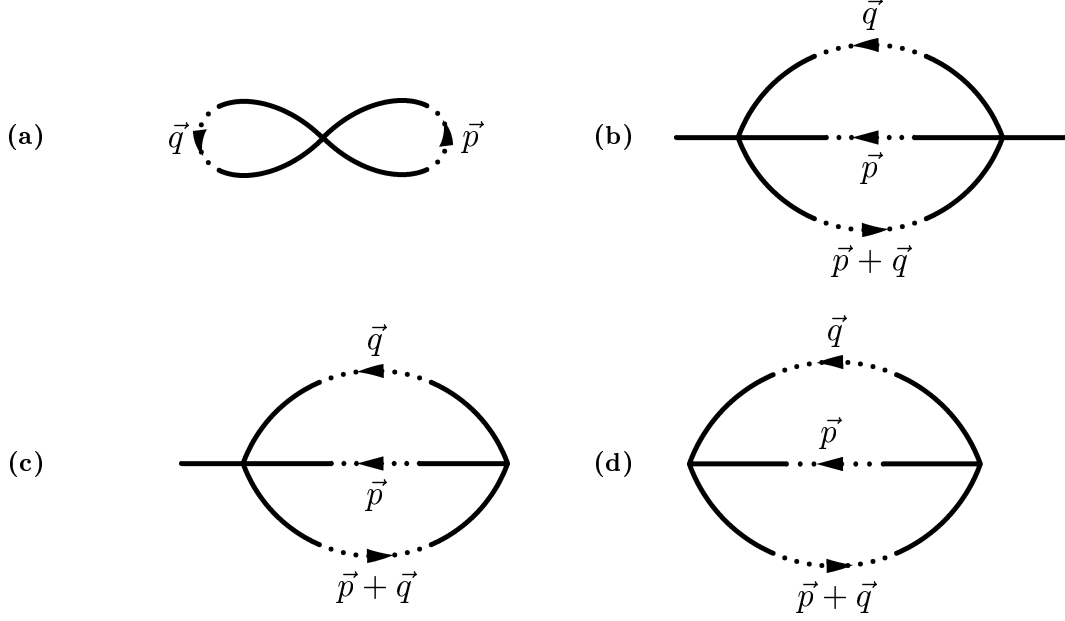


Figure 1: The four topologies appearing at two loops. Dots indicate parts of the graphs with additional external legs.

Now, for $t \rightarrow 0$ we have

$$\int_{\mathbf{p}} \frac{(\hat{p}^2)^a}{(\hat{p}^2 + t)^{a+b}} \sim \begin{cases} 1 & \text{for } b = 0 \\ \ln t & \text{for } b = 1 \\ t^{1-b} & \text{for } b \geq 2. \end{cases} \quad (80)$$

Therefore, for $t \rightarrow 0$, $u \rightarrow 0$ at fixed t/u , we have

$$t^n \chi_n / u \sim \begin{cases} u^{n/2-1} & \text{for } b = 0 \\ u^{n/2-1} \ln t & \text{for } b = 1 \\ u^{n/2-b} = u^{a/2} & \text{for } b \geq 2. \end{cases} \quad (81)$$

Thus, all contributions vanish except those with: (i) $n = 1$, $a = 1$, $b = 0$; (ii) $n = 2$, $a = 2$, $b = 0$; (iii) n even, $b = n/2$, $a = 0$. Contributions (iii) are those that appear in the standard theory without φ^3 interaction. Let us now show that contributions (i) and (ii) can be eliminated by redefining $r_c(u)$ and $h_c(u)$. Consider first χ_1 . At one loop we have

$$\frac{t}{u} \chi_1 = -\frac{h_c(u)}{u} - \frac{1}{2\sqrt{u}} \int_{\mathbf{p}} \frac{V_3(\mathbf{p})}{K(\mathbf{p}) + t} + O(\sqrt{u}). \quad (82)$$

For $t \rightarrow 0$ we have

$$\int_{\mathbf{p}} \frac{V_3(\mathbf{p})}{K(\mathbf{p}) + t} = \int_{\mathbf{p}} \frac{V_3(\mathbf{p})}{K(\mathbf{p})} + O(t \ln t). \quad (83)$$

Thus, if we define

$$h_c(u) = -\frac{1}{2}\sqrt{u} \int_{\mathbf{p}} \frac{V_3(\mathbf{p})}{K(\mathbf{p})}, \quad (84)$$

then $t\chi_1/u \sim O(t \ln t / \sqrt{u}, \sqrt{u}) \rightarrow 0$ in the critical crossover limit.

Now, let us consider the two-point function. At one loop we have

$$\frac{t^2}{u}\chi_2 \approx \frac{t}{u} - \frac{r_c(u)}{u} - \frac{1}{2} \int_{\mathbf{p}} \left[\frac{V_4(\mathbf{p})}{K(\mathbf{p}) + t} - \frac{V_3(\mathbf{p})^2}{(K(\mathbf{p}) + t)^2} \right]. \quad (85)$$

The first one-loop term is the contribution of the tadpole that has to be considered in the pure φ^4 theory and which requires an appropriate subtraction to scale correctly, cf. Eq. (76). The second one is due to the three-leg vertex and is finite for $t \rightarrow 0$. Therefore, if we define

$$r_c(u) = r_c^{\text{symm}}(u) + \frac{u}{2} \int_{\mathbf{p}} \frac{V_3(\mathbf{p})^2}{K(\mathbf{p})^2}, \quad (86)$$

where $r_c^{\text{symm}}(u)$ is given by Eq. (76), we cancel all contributions of the φ^3 vertex.

Now, let us repeat the same discussion at two loops, in order to understand the general mechanism. There are four different topologies that must be considered, see Fig. 1.

Topology (a).

A contribution to χ_n is proportional to

$$\frac{u^{n/2+1}}{t^n} \int_{\mathbf{p}} \int_{\mathbf{q}} V^{(4)}(\mathbf{p}, -\mathbf{p}, \mathbf{q}, -\mathbf{q}) \frac{V_4(\mathbf{p})^a V_4(\mathbf{q})^b V_3(\mathbf{p})^c V_3(\mathbf{q})^d}{(K(\mathbf{p}) + t)^{a+c+1} (K(\mathbf{q}) + t)^{b+d+1}}, \quad (87)$$

where $2a + 2b + c + d = n$. The leading contribution for $t \rightarrow 0$ is obtained by setting $V^{(4)}(\mathbf{p}, -\mathbf{p}, \mathbf{q}, -\mathbf{q}) \approx V^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = 1$. Then, the integral factorizes and we can use Eq. (80). Ignoring logarithmic terms, we see that the corresponding contribution to $t^n \chi_n / u$ scales as

$$\left(\frac{u}{t}\right)^{n/2} t^{n/2-a-b} = \left(\frac{u}{t}\right)^{n/2} t^{(c+d)/2}. \quad (88)$$

Thus, a nonvanishing contribution is obtained only for $c + d = 0$. Three-leg vertices can be neglected in the critical crossover limit.

Topology (b).

A contribution to χ_n has the form

$$\begin{aligned} & \frac{u^{n/2+1}}{t^n} \int_{\mathbf{p}} \int_{\mathbf{q}} [V^{(4)}(\mathbf{0}, \mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q})]^2 \\ & \times \frac{V_4(\mathbf{p})^a V_4(\mathbf{q})^b V_4(\mathbf{p} + \mathbf{q})^c V_3(\mathbf{p})^d V_3(\mathbf{q})^e V_3(\mathbf{p} + \mathbf{q})^f}{(K(\mathbf{p}) + t)^{a+d+1} (K(\mathbf{q}) + t)^{b+e+1} (K(\mathbf{p} + \mathbf{q}) + t)^{c+f+1}}, \end{aligned} \quad (89)$$

where $2(a + b + c) + d + e + f = n - 2$. The leading infrared contribution is obtained by approximating all expressions with their small- p behavior. Therefore, we can write for $t \rightarrow 0$

$$\frac{u^{n/2+1}}{t^n} \int_{\mathbf{p}} \int_{\mathbf{q}} \frac{(\hat{p}^2)^d (\hat{q}^2)^e (\widehat{p+q}^2)^f}{(\hat{p}^2 + t)^{a+d+1} (\hat{q}^2 + t)^{b+e+1} (\widehat{p+q}^2 + t)^{c+f+1}}. \quad (90)$$

Integrals of this type can be easily evaluated. By writing $\hat{p}^2 = (\hat{p}^2 + t) - t$ in the numerator, we obtain integrals

$$I_{mnr} \equiv \int_{\mathbf{p}} \int_{\mathbf{q}} \frac{1}{(\hat{p}^2 + t)^m (\hat{q}^2 + t)^n (\widehat{p+q}^2 + t)^r}, \quad (91)$$

with $m > 0$, $n > 0$, $r > 0$. Then, we can rescale $\mathbf{p} \rightarrow t^{1/2}\mathbf{p}$, $\mathbf{q} \rightarrow t^{1/2}\mathbf{q}$ and extend the integration over all \mathbb{R}^2 . This is possible since the corresponding continuum integral is finite. As a consequence the integral scales as $t^{2-m-n-r}$ and the corresponding contribution to $t^n \chi_n/u$ scales as

$$\left(\frac{u}{t}\right)^{n/2} t^{n/2-a-b-c-1} = \left(\frac{u}{t}\right)^{n/2} t^{(d+e+f)/2}. \quad (92)$$

Thus, a nonvanishing contribution is obtained only for $d+e+f=0$. Three-leg vertices can be neglected in the critical crossover limit.

Topology (c).

A contribution to χ_n has the form

$$\begin{aligned} & \frac{u^{n/2+1}}{t^n} \int_{\mathbf{p}} \int_{\mathbf{q}} V^{(4)}(\mathbf{0}, \mathbf{p}, \mathbf{q}, -\mathbf{p}-\mathbf{q}) V^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p}-\mathbf{q}) \\ & \times \frac{V_4(\mathbf{p})^a V_4(\mathbf{q})^b V_4(\mathbf{p}+\mathbf{q})^c V_3(\mathbf{p})^d V_3(\mathbf{q})^e V_3(\mathbf{p}+\mathbf{q})^f}{(K(\mathbf{p})+t)^{a+d+1} (K(\mathbf{q})+t)^{b+e+1} (K(\mathbf{p}+\mathbf{q})+t)^{c+f+1}}, \end{aligned} \quad (93)$$

where $2(a+b+c)+d+e+f=n-1$ and we assume without loss of generality $a \geq b \geq c$. We can now repeat the analysis performed for topology (b). We replace each quantity with its small-momentum behavior. In particular, we can replace $V^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p}-\mathbf{q})$ with $(\hat{p}^2 + \hat{q}^2 + \widehat{p+q}^2)$. Then, we rewrite each contribution in terms of the integrals I_{mnr} , cf. Eq. (91). However, in this case it is possible that one (and only one) of the indices vanishes. If this the case, I_{mnr} factorizes and we can use the one-loop result (80). A careful analysis shows that in all cases the integral scales as t^{-a-b-c} for $t \rightarrow 0$ except when $b=c=0$. In this case, if $a \neq 0$ the integral scales as $t^{-a} \ln t$, while for $a=0$ it scales as $\log^2 t$. Thus, ignoring logarithms all integrals scale as t^{-a-b-c} . Therefore, the corresponding contribution to $t^n \chi_n/u$ scales as

$$\left(\frac{u}{t}\right)^{n/2} t^{n/2-a-b-c} = \left(\frac{u}{t}\right)^{n/2} t^{(d+e+f+1)/2}. \quad (94)$$

These contributions always vanish.

Topology (d).

A contribution to χ_n has the form

$$\frac{u^{n/2+1}}{t^n} \int_{\mathbf{p}} \int_{\mathbf{q}} [V^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p}-\mathbf{q})]^2 \frac{V_4(\mathbf{p})^a V_4(\mathbf{q})^b V_4(\mathbf{p}+\mathbf{q})^c V_3(\mathbf{p})^d V_3(\mathbf{q})^e V_3(\mathbf{p}+\mathbf{q})^f}{(K(\mathbf{p})+t)^{a+d+1} (K(\mathbf{q})+t)^{b+e+1} (K(\mathbf{p}+\mathbf{q})+t)^{c+f+1}}, \quad (95)$$

where $2(a+b+c)+d+e+f=n$, $a \geq b \geq c$. We repeat the analysis done for topology (b) and (c). We find that the integral scales as

$$t^{-a-b-c+1} \begin{cases} \ln t/t & \text{for } a=b=c=0; \\ 1/t & \text{for } b=c=0, a \geq 1; \\ \ln t & \text{for } c=0, b=1, a \geq 1; \\ \text{constant} & \text{otherwise.} \end{cases} \quad (96)$$

Thus, if $b \neq 0$ and $c \neq 0$, ignoring logarithms, the contribution to $t^n \chi_n / u$ scales as

$$\left(\frac{u}{t}\right)^{n/2} t^{n/2-a-b-c+1} = \left(\frac{u}{t}\right)^{n/2} t^{(d+e+f)/2+1} \quad (97)$$

These contributions therefore always vanish in the critical crossover limit. If, however, $b = c = 0$, then

$$\left(\frac{u}{t}\right)^{n/2} t^{n/2-a-b-c} = \left(\frac{u}{t}\right)^{n/2} t^{(d+e+f)/2} \quad (98)$$

and thus one may have a finite (or a logarithmically divergent if $a = 0$) contribution for $d = e = f = 0$. Let us now focus on this last case in which $a = n/2 > 0$ with n even. Let us now show that these contributions are canceled by the one-loop counterterm due to $r_c(u)$. Indeed, at two loops we obtain a contribution to χ_n of the form

$$r_c(u) \frac{u^{n/2}}{t^n} \int_{\mathbf{p}} \frac{V_3^b(\mathbf{p}) V_4^a(\mathbf{p})}{[K(\mathbf{p}) + t]^{a+b+1}} = r_c(u) \frac{u^{n/2}}{t^n} \begin{cases} \ln t & \text{for } b = 0 \\ t^{-a} & \text{for } b \geq 1 \end{cases} \quad (99)$$

where $2a + b = n$. Thus, contributions to $t^n \chi_n / u$ scale as $t^{n/2-a} = t^{b/2}$ and thus vanish unless $b = 0$. Thus, for each even n there are two contributions that should be considered: one with topology (d) and one associated with the counterterm $r_c(u)$. Taking properly into account the combinatorial factors, their sum is given by (of course, we only consider here the contribution to $r_c(u)$ due to the three-leg vertices)

$$\frac{u^{n/2+1}}{t^n} \int_{\mathbf{p}} \frac{1}{(K(\mathbf{p}) + t)^{a+1}} \int_{\mathbf{q}} \left[\frac{V^{(3)}(\mathbf{p}, \mathbf{q}, -\mathbf{p} - \mathbf{q})^2}{(K(\mathbf{q}) + t)(K(\mathbf{p} + \mathbf{q}) + t)} - \frac{V^{(3)}(0, \mathbf{q}, -\mathbf{q})^2}{(K(\mathbf{q}) + t)^2} \right] \quad (100)$$

For $a \geq 1$, the subtracted term improves the infrared behavior, and indeed the integral scales as $t^{-a+1} \ln t$ and is therefore irrelevant in the critical crossover limit.

6.3 Higher powers of the fields

It is interesting to consider also Hamiltonians with higher powers of the field φ . The standard scaling argument indicates that these additional terms are irrelevant for the critical behavior, but in principle they can contribute to the renormalization constants like h_c and r_c . For this purpose let us suppose that the Hamiltonian contains also terms of the form

$$\Delta \mathcal{H}_{\text{eff}} = \sum_{k \geq 4} \frac{u^{(k-2)/2}}{k!} \int_{\mathbf{p}_1} \cdots \int_{\mathbf{p}_k} \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_k) V^{(k)}(\mathbf{p}_1, \dots, \mathbf{p}_k) \varphi(\mathbf{p}_1) \cdots \varphi(\mathbf{p}_k). \quad (101)$$

Note the particular dependence on the coupling constant u , which is motivated by the large- N calculation and is crucial in the argument reported below.

Let us consider the contributions of these additional terms. At tree level we have an additional contribution to χ_n given by

$$\Delta \chi_n = \frac{u^{(n-2)/2}}{t^n} V^{(n)}(\mathbf{0}, \dots, \mathbf{0}) \quad (102)$$

Thus, $t^n \Delta \chi_n / u \sim u^{(k-4)/2}$, that vanishes as $u \rightarrow 0$ for $k > 4$. At one-loop order there are additional contributions of the form

$$\frac{u^{n/2}}{t^n} \int_{\mathbf{p}} (K(\mathbf{p}) + t)^{-\sum a_i} V_3(\mathbf{p})^{a_3} \dots V_{n+2}(\mathbf{p})^{a_{n+2}}, \quad (103)$$

where $V_k(\mathbf{p}) = V^{(k)}(-\mathbf{p}, \mathbf{p}, \mathbf{0}, \dots, \mathbf{0})$ and $\sum_k (k-2)a_k = n$. Proceeding as in the previous Section, using Eq. (80), and noting that $\sum_{k>4} a_k > 0$ by hypothesis, we obtain

$$t^n \Delta \chi_n / u \sim u^{n/2-1} t^{1-\sum_{k>3} a_k} \sim u^{\frac{1}{2}a_3 + \frac{1}{2}\sum_{k\geq 5} (k-4)a_k}. \quad (104)$$

Here, possible logarithmic terms have been neglected. Then, since some a_k with $k \geq 5$ is nonvanishing by hypothesis, we find that this correction vanishes in the crossover limit. Therefore, no contribution survives at one loop. The same is expected at any perturbative order.

6.4 The general argument

The discussion reported above shows that at two loops one can define a critical crossover limit with crossover functions that are identical to those of the symmetric theory. This is expected to be a general result since formally the added interaction is irrelevant. This result can be understood diagrammatically.

Consider the continuum theory with Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{\mu} (\partial_{\mu} \phi)^2 + \frac{t}{2} \phi^2 + \frac{\sqrt{u}}{3!} \phi^2 \square \phi + \frac{u}{4!} \phi^4. \quad (105)$$

Given an l -loop diagram contributing to the zero-momentum n -point irreducible correlation function, we can write the corresponding Feynman integral D as

$$D \sim u^{N_4 + N_3/2} \int_{\mathbf{p}_1} \dots \int_{\mathbf{p}_l} I(\mathbf{p}_1, \dots, \mathbf{p}_l) \quad (106)$$

where N_3 and N_4 are the number of three-leg and four-leg vertices respectively. Using the topological relations $N_i = N_3 + N_4 + l - 1$ and $N_3 + 2N_4 - 2l + 2 = n$, where N_i is the number of internal lines, it is easy to see that

$$I(\lambda \mathbf{p}_1, \dots, \lambda \mathbf{p}_l) = \lambda^{2(1-N_4)} I(\mathbf{p}_1, \dots, \mathbf{p}_l). \quad (107)$$

Thus, there are primitively divergent diagrams for any n : those with $N_4 = 0$ (and correspondingly $N_3 = n + 2l - 2$) are quadratically divergent, while those with $N_4 = 1$ (and $N_3 = n + 2l - 4$) are logarithmically divergent. Analogously, diagrams contributing to the second derivatives of the n -point irreducible correlation functions with respect to the external momenta diverge logarithmically when $N_4 = 0$. Thus, the renormalized Hamiltonian contains an infinite number of counterterms and is given by

$$\mathcal{H}^{\text{ren}} = \mathcal{H} + \sum_n [Z_n(\Lambda) \phi^n + \zeta_n(\Lambda) \phi^{n-1} \square \phi], \quad (108)$$

where Λ is a generic cutoff (in our explicit calculation Λ is the inverse lattice spacing $1/a$, that never explicitly appears in the calculation since we set $a = 1$). Now, let us show that all counterterms except those computed in the previous Section can be neglected in the critical crossover limit. Suppose that we wish to compute $Z_n(\Lambda)$ at l loops. Keeping into account that the divergence may be quadratic or logarithmic, we expect the l -loop divergent contribution to χ_n to be of the form

$$\frac{u^{n/2+l-1}}{t^n} (a_1 \Lambda^2 + t P_1[\ln(\Lambda^2/t)] + P_2[\ln(\Lambda^2/t)]), \quad (109)$$

where $P_1(x)$ and $P_2(x)$ are polynomials and a_1 a constant. Therefore, the contribution to $t^n \chi_n/u$ vanishes unless $n/2 + l - 2 \leq 0$. The only two cases satisfying this condition (of course $n \geq 1$ and $l \geq 1$) are $l = n = 1$, $l = 1$ and $n = 2$, which are the cases considered before. Let us now consider the contributions to $\bar{\chi}_{n,i}$, which is the first derivative of the n -point connected correlation function with respect to the square of an external momentum \mathbf{p} computed at zero momentum. Since momenta scale as $t^{1/2}$, in the critical crossover limit we should have $\bar{\chi}_{n,i} \approx u t^{-n-1} \bar{f}_n(t/u, H/u)$. The divergent contributions are logarithmic (diagrams with $N_4 = 0$) and therefore we expect an l -loop contribution of the form

$$\frac{u^{n/2+l-1}}{t^n} (P[\ln(\Lambda^2/t)] + \text{finite terms}). \quad (110)$$

Considering $t^{n+1} \bar{\chi}_{n,i}/u$, we see that this contribution always vanishes in the critical crossover limit. Therefore, the renormalization constants $\zeta_n(\Lambda)$ can be neglected. Thus, the only renormalizations needed are those that we have considered. Finally, let us show that correlation functions computed in the renormalized theory have the correct scaling behavior. Indeed, in the renormalized theory diagrams scale canonically with possible logarithmic corrections. Therefore, D defined in Eq. (106) scales as $u^{N_4+N_3/2} t^{1-N_4} \times \text{logs}$, so that the contribution of D to $t^n \chi_n/u$ scales as

$$\frac{t^n}{u} \times \frac{1}{t^n} \times u^{N_4+N_3/2} t^{1-N_4} \sim u^{N_3/2}. \quad (111)$$

Therefore, the only nonvanishing diagrams have $N_3 = 0$, confirming the claim that three-leg vertices do not play any role.

6.5 A unique definition for the renormalization functions $h_c(u)$ and $r_c(u)$

In this Section we wish to discuss again the definition of $r_c(u)$ and $h_c(u)$. It is obvious that these functions are not uniquely defined, since one can add a term proportional to u without modifying the scaling behavior. We wish now to fix this ambiguity by requiring that $t = h = 0$ corresponds to the critical point.

It is easy to see that no modifications are needed for $h_c(u)$. Indeed, with the choice (84) one obtains the correlation functions of the symmetric theory and in this case the critical point is uniquely defined by $h = 0$ by symmetry. The proper definition of $r_c(u)$ requires more care, since we must perform a nonperturbative calculation in order to identify the critical point. For this purpose we will use the fact that in the critical crossover limit the

perturbative expansion in powers of u is equivalent to the perturbative expansion in the continuum ϕ^4 theory once a proper mass renormalization is performed.

In the continuum theory, if $\tilde{t} \equiv t_{\text{cont}}/u_{\text{cont}}$ is the adimensional reduced temperature defined so that $\tilde{t} = 0$ corresponds to the critical point, we have at one loop, cf. Eq. (2.10) of Ref. [29],

$$u_{\text{cont}}\chi_{2,\text{cont}} = \frac{1}{\tilde{t}} + \frac{1}{8\pi\tilde{t}^2} \left(\ln \frac{8\pi\tilde{t}}{3} + 3 + 8\pi D_2 \right) + O(\tilde{t}^{-3} \ln \tilde{t}^2), \quad (112)$$

where D_2 is a nonperturbative constant that can be expressed in terms of renormalization-group functions, cf. Eq. (2.11) of Ref. [29]. By using the four-loop perturbative results of Ref. [32], Ref. [29] obtained the estimate $D_2 = -0.0524(2)$. It is not clear whether the error can really be trusted, since in two dimensions the resummation of the perturbative expansion is not well behaved due to nonanalyticities of the renormalization-group functions at the fixed point [33, 34]; still, the estimate should provide the correct order of magnitude.

The expansion (112) should be compared with the perturbative expansion of χ_2 in the lattice model. We write $r_c(u)$ as

$$r_c(u) = \frac{u}{8\pi} \ln u + \frac{u}{2} \int_{\mathbf{p}} \frac{V_3(\mathbf{p})^2}{K(\mathbf{p})^2} + Au, \quad (113)$$

where A is a constant to be determined. From Eq. (85) we obtain

$$u\chi_2 = \frac{u}{t} + \frac{u^2}{8\pi t^2} \left\{ \log \frac{t}{32u} - 8\pi A - 4\pi \int_{\mathbf{p}} \left[\frac{V_4(\mathbf{p})}{K(\mathbf{p})} - \frac{1}{\hat{p}^2} \right] \right\} + O(u^3). \quad (114)$$

By comparing this result with Eq. (112) we obtain

$$A = -D_2 - \frac{1}{8\pi} \ln \frac{256\pi}{3} - \frac{3}{8\pi} - \frac{1}{2} \int_{\mathbf{p}} \left[\frac{V_4(\mathbf{p})}{K(\mathbf{p})} - \frac{1}{\hat{p}^2} \right]. \quad (115)$$

If we use definition (113) with A fixed by Eq. (115), the critical point corresponds to $\tilde{t} = 0$.

6.6 Critical crossover limit in the large- N case

In Sec. 5 we showed that the effective Hamiltonian has the form (71) with $u \sim 1/N$. However, in the large- N case the parameters that can be tuned are $\Delta_m \equiv m_0^2 - m_{0c}^2$ and $\Delta_p \equiv p - p_c$, and moreover all quantities, beside r and H , depend on these two variables. We wish now to determine the changes, if any, that appear in the previous treatment due to the dependence of $V^{(k)}(\mathbf{p}_1, \dots, \mathbf{p}_k)$ and $K(\mathbf{p})$ on Δ_m and Δ_p . As we show in the next Section, Δ_m and Δ_p scale respectively as $1/N^{1/3}$ and $1/N$, so that we can assume an additional dependence on $u^{1/3}$, i.e. we consider $K(\mathbf{p}; u^{1/3})$ and $V^{(k)}(\mathbf{p}_1, \dots, \mathbf{p}_k; u^{1/3})$. Note that, by definition, $K(\mathbf{0}; u^{1/3}) = 0$ for all values of u . Moreover, we assume, as in the case of interest, that $V^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}; u^{1/3}) = 0$ for all values of u . Following the calculation presented in Sec. 6.2, it is easy to realize that the dependence of the vertices on u is irrelevant except in $h_c(u)$. In this case, Eq. (82) becomes

$$\frac{t}{u} \chi_1 = -\frac{h_c(u)}{u} - \frac{1}{2\sqrt{u}} \int_{\mathbf{p}} \frac{V_3(\mathbf{p}; u^{1/3})}{K(\mathbf{p}; u^{1/3}) + t} + O(\sqrt{u}). \quad (116)$$

Because of the prefactor $1/\sqrt{u}$ we must take here into account the first correction proportional to $u^{1/3}$. Thus, if $V_3(\mathbf{p}; u^{1/3}) \approx V_{3,0}(\mathbf{p}) + u^{1/3}V_{3,1}(\mathbf{p})$ and $K(\mathbf{p}; u^{1/3}) \approx K_0(\mathbf{p}) + u^{1/3}K_1(\mathbf{p})$, we obtain

$$\frac{t}{u}\chi_1 = -\frac{h_c(u)}{u} - \frac{1}{2\sqrt{u}} \int_{\mathbf{p}} \left[\frac{V_{3,0}(\mathbf{p}) + u^{1/3}V_{3,1}(\mathbf{p})}{K_0(\mathbf{p}) + t} - \frac{u^{1/3}V_{3,0}(\mathbf{p})K_1(\mathbf{p})}{(K_0(\mathbf{p}) + t)^2} \right] + O(u^{1/6}). \quad (117)$$

Now, t can be set to zero without generating infrared divergences, neglecting corrections of order $u^{-1/2}t \ln t \sim u^{1/2} \ln u$. It follows

$$h_c(u) = -\frac{\sqrt{u}}{2} \int_{\mathbf{p}} \left[\frac{V_{3,0}(\mathbf{p})}{K_0(\mathbf{p})} + u^{1/3} \frac{V_{3,1}(\mathbf{p})K_0(\mathbf{p}) - V_{3,0}(\mathbf{p})K_1(\mathbf{p})}{K_0(\mathbf{p})^2} \right]. \quad (118)$$

Eq. (118) represents the only equation in which the explicit dependence of the vertices on Δ_m should be considered (Δ_p is proportional to u and thus it can always be set to zero). In all other cases, we can simply set $\Delta_m = \Delta_p = 0$.

7 Crossover between mean-field and Ising behavior

In this Section we wish to apply the above-reported results for the critical crossover limit to the Hamiltonian (68) of the zero mode. If we identify

$$u \equiv \frac{1}{N} \bar{V}^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}), \quad (119)$$

the effective Hamiltonian (52) corresponds to the Hamiltonian discussed in Sec. 6, apart from a rescaling of the three-leg and four-leg vertices.

We begin by defining the additive renormalization constants. The mass renormalization constant is given by, cf. Eqs. (113) and (115),

$$\begin{aligned} r_c(N) = & \frac{u}{8\pi} \ln \left(\frac{3u}{256\pi} \right) - \frac{u}{8\pi} (3 + 8\pi D_2) \\ & + \frac{u}{2} \int_{\mathbf{p}} \left\{ \frac{[\bar{V}^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p})]^2}{\bar{V}^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})K(\mathbf{p})^2} - \frac{\bar{V}^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p})}{\bar{V}^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})K(\mathbf{p})} + \frac{1}{\hat{p}^2} \right\}, \end{aligned} \quad (120)$$

where D_2 is a nonperturbative constant defined in Ref. [29] (numerically $D_2 \approx -0.052$). As discussed in Sec. 6.6, we can compute all quantities appearing in Eq. (120) at the critical point and keep only the leading terms for $N \rightarrow \infty$. We thus obtain

$$\begin{aligned} r_c(N) = & \frac{u}{8\pi} \ln \left(\frac{3u}{256\pi} \right) - \frac{u}{8\pi} (3 + 8\pi D_2) \\ & + \frac{\alpha^2}{2N} \int_{\mathbf{p}} \left\{ [\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p})\tilde{P}_0(\mathbf{p})]^2 - \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p})\tilde{P}_0(\mathbf{p}) + \frac{\alpha^2}{\hat{p}^2} \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \right\}, \end{aligned} \quad (121)$$

where all quantities are computed at the critical point.

Analogously, we should introduce a counterterm for the magnetic field:

$$h_c(N) = -\frac{1}{2}\sqrt{u} \int_{\mathbf{p}} \frac{\bar{V}^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p})}{[\bar{V}^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})]^{1/2}K(\mathbf{p})}. \quad (122)$$

As discussed in Sec. 6.6, such a quantity should not be simply computed at the critical point, but one should also take into account the additional corrections of order $N^{-1/3}$. Since $k \sim O(N^{1/6})$ in the critical crossover limit, we have

$$h_c(N) = -\frac{\alpha}{2\sqrt{N}} \int_{\mathbf{p}} \left[\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p}) - \frac{\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})}{\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})} \tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{p}, -\mathbf{p}) \right] \frac{1}{\tilde{P}_0^{-1}(\mathbf{p}) - \tilde{P}_0^{-1}(\mathbf{0})} - \frac{\alpha}{2\sqrt{N}} \frac{\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})}{\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})} \int_{\mathbf{p}} [\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{p}, -\mathbf{p}) \tilde{P}_0(\mathbf{p})]^2. \quad (123)$$

The second integral should be computed at the critical point, while the first one and the prefactor of the second one should be expanded around the critical point. Since, as we shall show below, $\Delta_m \equiv m_0^2 - m_{0c}^2 \sim N^{-1/3}$ and $p - p_c \sim N^{-1}$, it is enough to compute the first correction in Δ_m . In practice, we find that the renormalization terms have the form

$$r_c(N) = \frac{1}{N}(r_{c0} \ln N + r_{c1}),$$

$$h_c(N) = \frac{1}{\sqrt{N}}(h_{c0} + \Delta_m h_{c1}). \quad (124)$$

Once $r_c(N)$ and $h_c(N)$ are computed, we can define the scaling variables $x_t \sim t/u$ and $x_h \sim h/u$. Choosing the normalizations appropriately for later convenience, we define

$$x_t \equiv \frac{N}{\alpha^2} [\bar{P}^{-1}(0) - r_c(N)],$$

$$x_h \equiv \frac{N}{\alpha} [H - h_c(N)]. \quad (125)$$

Now, the critical crossover limit is obtained by tuning p and m_0^2 around the critical point in such a way that x_t and x_h are kept constant, i.e. $p \rightarrow p_{\text{crit}}$, $m_0^2 \rightarrow m_{0,\text{crit}}^2$, $N \rightarrow \infty$ at fixed x_t and x_h . Note that here p_{crit} and $m_{0,\text{crit}}^2$ correspond to the position of the critical point as a function of N (thus $p_{\text{crit}} \rightarrow p_c$ and $m_{0,\text{crit}}^2 \rightarrow m_{0c}^2$ as $N \rightarrow \infty$) and are obtained by requiring $x_t = x_h = 0$ (cf. Sec. 6.5).

In order to compute the relation between p , m_0^2 and x_t , x_h we set $\Delta_m \equiv m_0^2 - m_{0c}^2$ and $\Delta_p \equiv p - p_c$ and expand Eq. (125) in powers of Δ_m and Δ_p . In the following we shall show that $\Delta_m \sim N^{-1/3}$ and $\Delta_p \sim N^{-1}$, so that the relevant terms are

$$x_t = N(b_{t0}\Delta_p + b_{t1}\Delta_p\Delta_m + b_{t2}\Delta_m^3) + d_1 \ln N + d_2 \quad (126)$$

$$x_h = N^{3/2}(b_{h0}\Delta_p^2 + b_{h1}\Delta_p\Delta_m + b_{h2}\Delta_p\Delta_m^2 + b_{h3}\Delta_m^3 + b_{h4}\Delta_m^4) + N^{1/2}(d_{30} + d_{31}\Delta_m). \quad (127)$$

The constants are obtained by expanding $\bar{P}^{-1}(\mathbf{0})$ and H around the critical point and by using the expansions (124). All quantities are analytic in Δ_m and Δ_p and several terms are absent because of identities (60), (61), and (62). In particular, a term proportional to Δ_m^2 is absent in the equation for x_t . This is a consequence of Eq. (62). Indeed, we have

$$\frac{1}{\alpha^2} \bar{P}^{-1}(\mathbf{0}) = \tilde{P}_0^{-1}(\mathbf{0}) - \frac{[\tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0})]^2}{2\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})} + O(\Delta_m^3, N^{-1})$$

$$\begin{aligned}
&= \frac{1}{2} \left[\frac{\partial^2 \tilde{P}_0^{-1}(\mathbf{0})}{\partial(m_0^2)^2} - \frac{1}{\tilde{V}_0^{(4)}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})} \left(\frac{\partial}{\partial m_0^2} \tilde{V}_0^{(3)}(\mathbf{0}, \mathbf{0}, \mathbf{0}) \right)^2 \right] \Delta_m^2 + O(\Delta_m^3, N^{-1}) \\
&= 0 + O(\Delta_m^3, N^{-1}),
\end{aligned} \tag{128}$$

where in the last step we have used Eq. (62).

We wish now to determine the behavior of Δ_m and Δ_p that is fixed by Eqs. (126) and (127). We assume

$$\Delta_m = \frac{\delta_{m0}}{N^\alpha}, \quad \Delta_p = \frac{\delta_{p0}}{N^\beta}, \tag{129}$$

where δ_{m0} and δ_{p0} are nonvanishing constants and α and β exponents to be determined. If $\beta < 1$, Eq. (126) implies

$$b_{t0}\delta_{p0}N^{1-\beta} + b_{t2}\delta_{m0}N^{1-3\alpha} = o(N^{1-\beta}), \tag{130}$$

which requires $\beta = 3\alpha$. Now, consider Eq. (127). The term $N^{3/2}\Delta_m^3$ is of order $N^{3/2-\alpha}$ and cannot be made to vanish. Therefore, we must have $\beta \geq 1$. Considering again Eq. (127), it is easy to see that all terms containing Δ_p cannot increase as fast as $N^{1/2}$. Therefore, cancellation of the term $d_{30}N^{1/2}$ requires $\alpha = 1/3$. Consideration of Eq. (126) implies finally $\beta = 1$. This analysis can be extended to the subleading corrections, obtaining an expansion of the form

$$\begin{aligned}
\Delta_m &= \frac{\delta_{m0}}{N^{1/3}} + \frac{\delta_{m1}}{N^{2/3}} + \frac{\delta_{m2}}{N^{5/6}} + O(N^{-1}) \\
\Delta_p &= \frac{\delta_{p0}}{N} + \frac{\delta_{p1}}{N^{7/6}} + O(N^{-4/3}).
\end{aligned} \tag{131}$$

The coefficients are given by

$$\begin{aligned}
\delta_{m0} &= - \left(\frac{d_{30}}{b_{h3}} \right)^{1/3}, \\
\delta_{m1} &= \frac{\delta_{m0}^2}{3b_{t0}d_{30}} [b_{t0}d_{31} - b_{h1}(d_1 \ln N + d_2)] + \frac{b_{h1}\delta_{m0}^2}{3b_{t0}d_{30}} x_t + \\
&\quad + \frac{\delta_{m0}^5}{3b_{t0}d_{30}} (b_{h4}b_{t0} - b_{h1}b_{t2}), \\
\delta_{p0} &= -\frac{1}{b_{t0}} (\delta_{m0}^3 b_{t2} + d_1 \ln N + d_2 - x_t).
\end{aligned} \tag{132}$$

We are not able to compute δ_{m2} and δ_{p1} but we can however compute a relation between these two quantities. We obtain

$$\delta_{p1} = -\frac{3\delta_{m0}\delta_{m2}b_{h3}}{b_{h1}} + \frac{1}{b_{h1}\delta_{m0}} x_h. \tag{133}$$

Correspondingly, by using Eq. (13), we can compute the expansion of u_h :

$$u_h = \frac{u_{h0}}{N} + \frac{u_{h1}}{N^{4/3}} + \frac{u_{h2}}{N^{3/2}}, \tag{134}$$

where u_{h1} depends on x_t and u_{h2} on x_h . We thus define a new scaling field by requiring that no term proportional to $x_t N^{-4/3}$ is present. For this purpose we set

$$\hat{u}_h = u_h + \frac{x_{\text{mix}}}{N^{1/3}}(p - p_c) \quad (135)$$

where the coefficient x_{mix} is given by

$$x_{\text{mix}} = \frac{(a_{03}b_{h1} - a_{11}b_{h3})\delta_{m0}}{b_{h3}}. \quad (136)$$

The new scaling field has an expansion of the form

$$\hat{u}_h = \frac{\hat{u}_{h0}}{N} + \frac{\hat{u}_{h1}}{N^{4/3}} + \frac{\hat{u}_{h2}}{N^{3/2}}, \quad (137)$$

where

$$\hat{u}_{h0} = a_{03}\delta_{m0}^3, \quad (138)$$

$$\hat{u}_{h1} = \frac{\delta_{m0}}{b_{h3}}[-a_{03}d_{31} + (a_{04}b_{h3} - a_{03}b_{h4})\delta_{m0}^3] \quad (139)$$

$$\hat{u}_{h2} = \frac{a_{03}}{b_{h3}}x_h. \quad (140)$$

Interestingly enough, in \hat{u}_{h2} all terms proportional to the unknown quantity δ_{m2} cancel. At this point we can easily compute the $1/N$ expansion of the critical point p_{crit} , β_{crit} . It is enough to set $x_h = x_t = 0$ in the previous expansions, obtaining

$$p_{\text{crit}} = p_c + \frac{\delta_{p0}}{N} \Big|_{x_t=0} + O(N^{-7/6}), \quad (141)$$

$$\beta_{\text{crit}} = \beta_c + \frac{\hat{u}_{h0} + a_{10}\delta_{p0}|_{x_t=0}}{N} + O(N^{-7/6}). \quad (142)$$

It follows that

$$p - p_{\text{crit}} \approx \left(-\frac{2}{3} \frac{b_{t1}b_{h1}\delta_{m0}^2}{d_{30}} + 1 \right) \frac{x_t}{b_{t0}N}, \quad (143)$$

$$\hat{u}_h - \hat{u}_{h,\text{crit}} \approx \frac{3a_{03}b_{t0} - a_{11}b_{t1}}{3b_{h3}b_{t0} - b_{h1}b_{t1}} \frac{x_h}{N^{3/2}}, \quad (144)$$

where $\hat{u}_{h,\text{crit}}$ is the value of \hat{u}_h at the critical point. Therefore, the $1/N$ corrections modify the position of the critical point and change the magnetic scaling field which should now be identified with $\bar{u}_h = \hat{u}_h - \hat{u}_{h,\text{crit}}$ (as we already discussed the thermal magnetic field is not uniquely defined and we take again $p - p_{\text{crit}}$). Eqs. (143) and (144) also indicate which are the correct scaling variables. Ising behavior is observed only if $N(p - p_{\text{crit}})$ and $N^{3/2}(\hat{u}_h - \hat{u}_{h,\text{crit}})$ are both small; in the opposite case mean-field behavior is observed. Therefore, as N increases the width of the critical region decreases, and no Ising behavior is observed at $N = \infty$ exactly.

8 Critical behavior of $\langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} \rangle$

In this Section we wish to compute the large- N behavior of

$$\overline{E} = \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} \rangle. \quad (145)$$

Such a quantity does not coincide with the energy. However, as far as the critical behavior is concerned, there should not be any significant difference. In order to perform the computation, note that the equations of motion for the field $\lambda_{x\mu}$ give

$$\langle \rho_{x\mu} \rangle = 1 + \overline{E}. \quad (146)$$

Thus, we have

$$\overline{E} = \tau - 1 + \frac{1}{\sqrt{N}} \langle \hat{\rho}_{x\mu} \rangle = \tau - 1 + \frac{1}{\sqrt{N}} \sum_B U_{4B}(\mathbf{0}) \langle \Phi_{Bx} \rangle. \quad (147)$$

Therefore, we need to compute the correlations $\langle \phi \rangle$ and $\langle \varphi_a \rangle$. For the zero mode we have immediately

$$\langle \phi_x \rangle = \alpha \langle \chi_x \rangle - k = \frac{\alpha}{x_t} f_1^{\text{symm}}(x_t, x_h) - k, \quad (148)$$

where $f_1^{\text{symm}}(x_t, x_h)$ is the crossover function for the magnetization in the Ising model and the constant k diverges as $N^{1/6}$, modulo some obvious normalizations.

Let us now consider $\langle \varphi_a \rangle$ and show that such a correlation vanishes as $N \rightarrow \infty$ in the critical crossover limit. Indeed, we can write

$$\begin{aligned} \langle \varphi_a \rangle &= \frac{1}{\sqrt{N}} \hat{P}_{ab}(\mathbf{0}) \int_{\mathbf{p}} [\hat{V}_{bcd}(\mathbf{0}, \mathbf{p}, -\mathbf{p}) \hat{P}_{cd}(\mathbf{p}) + (\hat{V}_{b11}(\mathbf{0}, \mathbf{p}, -\mathbf{p}) - \hat{V}_{b11}(\mathbf{0}, \mathbf{0}, \mathbf{0})) \hat{P}_{11}(\mathbf{p})] \\ &\quad + \frac{1}{\sqrt{N}} \hat{P}_{ab}(\mathbf{0}) \hat{V}_{b11}(\mathbf{0}, \mathbf{0}, \mathbf{0}) \langle \phi_x^2 \rangle. \end{aligned} \quad (149)$$

Note that in the last term we have replaced

$$\int_{\mathbf{p}} \hat{P}_{11}(\mathbf{p}) = \langle \phi_x^2 \rangle, \quad (150)$$

a necessary step, since the integral diverges at the critical point and therefore should be computed in the effective theory for the zero mode. Now, we have

$$\langle \phi_x^2 \rangle = k^2 - 2\alpha k \langle \chi_x \rangle + \alpha^2 \langle \chi_x^2 \rangle. \quad (151)$$

In the critical crossover limit, the two expectation values are replaced by the crossover functions for the magnetization and the energy and by a regular term. Therefore, for $N \rightarrow \infty$, the leading behavior is $\langle \phi_x^2 \rangle = k^2 \sim N^{1/3}$. It follows that $\langle \varphi_a \rangle \sim N^{-1/6}$. In conclusion we can write

$$\begin{aligned} \overline{E} &= \tau - 1 + \frac{1}{\sqrt{N}} U_{41}(\mathbf{0}) \left[\frac{\alpha}{x_t} f_1^{\text{symm}}(x_t, x_h) - k \right] \\ &= \overline{E}_{\text{reg}} + \frac{1}{\sqrt{N}} U_{41}(\mathbf{0}) \frac{\alpha}{x_t} f_1^{\text{symm}}(x_t, x_h) + O(N^{-2/3}), \end{aligned} \quad (152)$$

where $\overline{E}_{\text{reg}}$ is the regular part of \overline{E} :

$$\begin{aligned}\overline{E}_{\text{reg}} &= \tau - 1 - \frac{1}{\sqrt{N}} U_{41}(\mathbf{0})k \\ &= \tau_0 - 1 + [\tau_1 - k_0 U_{41}(\mathbf{0})] \delta_{m0} N^{-1/3} + O(N^{-2/3}),\end{aligned}\tag{153}$$

where we have written $\tau \approx \tau_0 + \tau_1 \Delta_m$ and $k \approx k_0 \Delta_m \sqrt{N}$.

Equation (152) shows that the singular part of \overline{E} behaves as the magnetization in the Ising model. For $x_t \ll 1$ and $x_h \ll 1$ one observes Ising behavior and thus

$$\begin{aligned}\overline{E} - \overline{E}_{\text{reg}} &\sim |x_t|^{\beta_I} && \text{for } x_h = 0, \text{ low } t \text{ phase} \\ \overline{E} - \overline{E}_{\text{reg}} &\sim |x_h|^{1/\delta_I} && \text{for } x_h \neq 0,\end{aligned}\tag{154}$$

where $\beta_I = 1/8$ and $\delta_I = 15$. On the other hand, in the opposite limit we have

$$\begin{aligned}\overline{E} - \overline{E}_{\text{reg}} &\sim |x_t|^{\beta_{MF}} && \text{for } x_h = 0, |x_t| \rightarrow \infty, \text{ low } t \text{ phase} \\ \overline{E} - \overline{E}_{\text{reg}} &\sim |x_h|^{1/\delta_{MF}} && \text{for } |x_h| \rightarrow \infty\end{aligned}\tag{155}$$

with $\beta_{MF} = 1/2$ and $\delta_{MF} = 3$. Note that the limit $|x_t| \rightarrow \infty$ and $|x_h| \rightarrow \infty$ should always be taken close to the critical limit. Therefore, $|x_h| \rightarrow \infty$ means that we should consider $N \rightarrow \infty$, $\hat{u}_h \rightarrow \hat{u}_{h,\text{crit}}$, $p \rightarrow p_{\text{crit}}$ in such a way that $N^{3/2}(\hat{u}_h - \hat{u}_{h,\text{crit}}) \rightarrow \infty$, i.e. N should increase much faster than the rate of approach to the critical point.

9 Numerical results for selected Hamiltonians

In this Section we present some numerical results for some selected Hamiltonians. First, as in Ref. [21], we consider

$$W(x) = \frac{2}{p} \left(\frac{x}{2}\right)^p.\tag{156}$$

Second, we consider the mixed $O(N)$ - RP^{N-1} model with Hamiltonian [20]

$$\mathcal{H} = -N\beta_V \sum_{x\mu} (\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu}) - \frac{N\beta_T}{2} \sum_{x\mu} (\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu})^2.\tag{157}$$

This Hamiltonian corresponds to the function

$$W(x) = px + \frac{1}{4}(1-p)x^2,\tag{158}$$

where we set $\beta_V = (1+p)\beta/2$ and $\beta_T = (1-p)\beta/2$. This Hamiltonian is ferromagnetic for $p > -1$. Note that for $p = -1$ we obtain the RP^{N-1} Hamiltonian [20, 24, 25, 35–41]

$$\mathcal{H} = -\frac{N\beta}{2} \sum_{x\mu} (\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu})^2,\tag{159}$$

	\mathcal{H}_1	\mathcal{H}_2
β_c	1.334721915850	0.9181906464057
p_c	4.537856778637	-0.9707166650184
m_{0c}^2	0.1501849439193	0.8657494320430
a_{10}	0.4359516292302	-1.1359750388653
a_{11}	0.5042522341176	-0.6248171602172
a_{12}	-1.2793594495686	0.1122363376128
a_{03}	-1.3015714087645	-0.0061080440602
a_{04}	67.512019516378	0.2847340288532
b_{t0}	-0.04261881236908	0.21169346791995
b_{t1}	-0.0043954143923	0.05334711842197
b_{t2}	1.85335235385232	0.00015229016444
b_{h0}	-0.0077917231312	-0.664558775414698
b_{h1}	0.085888253027	-0.094120307536470
b_{h2}	-0.5785580673083	0.049596987743194
b_{h3}	-0.221693999401	-0.000920094744508
d_1	0.205	0.2678
d_2	0.614	-0.808514
d_{30}	0.374	1.1692
α^{-2}	0.0315969	0.00933367

Table 1: Numerical estimates for Hamiltonian (156) (\mathcal{H}_1) and (158) (\mathcal{H}_2).

that has the additional gauge invariance $\boldsymbol{\sigma}_x \rightarrow \epsilon_x \boldsymbol{\sigma}_x$, $\epsilon_x = \pm 1$. Under standard assumptions, the large- N analysis should apply also to this last model: the local gauge invariance should not play any role.²

In the large- N limit [22], the first Hamiltonian has a critical point at $\beta_c \approx 1.335$ and $p_c \approx 4.538$. By using the numerical results reported in Table 1 we can compute the first corrections to the critical parameters. We obtain

$$\beta_c \approx 1.335 + \frac{1}{N} (36.127 + 2.093 \ln N), \quad (160)$$

$$p_c \approx 4.538 + \frac{1}{N} (87.92 + 4.80 \ln N). \quad (161)$$

Note the presence of a $\ln N/N$ correction due to the nonanalytic nature of the renormalization counterterms. The correction terms are quite large, indicating that the large- N results are quantitatively predictive only for large values of N . This is not totally unexpected since [21] $p_c \approx 20$ for $N = 3$, that is quite far from the large- N estimate $p_c \approx 4.538$. If we substitute $N = 3$ in Eq. (161), we obtain $p_c \approx 35$, which shows that the corrections have the correct sign and give at least the correct order of magnitude.

²The irrelevance of the \mathbb{Z}_2 symmetry for the large- β behavior of RP^{N-1} models have been discussed in detail in Ref. [24, 40, 41]. Thus, in spite of the additional local invariance, RP^{N-1} models are expected to be asymptotically free and to be described by the perturbative renormalization group.

For $N = \infty$ the second Hamiltonian has a critical point at [20] $\beta_c \approx 0.918$ and $p_c \approx -0.971$. Including the first correction we obtain

$$\beta_c = 0.9182 - \frac{1}{N} (11.062 - 1.437 \ln N), \quad (162)$$

$$p_c = -0.9707 + \frac{1}{N} (2.905 - 1.265 \ln N). \quad (163)$$

In this case the corrections are smaller. However, for $N \leq 99.4$ they predict $p_c < -1$, although only slightly. This is of course not possible, since for $p < -1$ the system is no longer ferromagnetic. Thus, we expect the transition to disappear for some $N = N_c$, with $N_c \approx 100$ (this is of course a very rough estimate). Since $p_c \approx -1$, we also predict RP^{N-1} models to show a very weak first-order transition for large N .

10 Conclusions

In this paper we investigate generic one-parameter families of N -vector models that show a line of first-order finite- β transitions. We focus on the endpoint of the first-order transition line where energy-energy correlations become critical, while the spin-spin correlation length remains finite, in agreement with Mermin-Wagner theorem [1]. We show that, at the critical point, the standard $1/N$ expansion breaks down, since the inverse propagator of the auxiliary fields has a zero eigenvalue. A careful treatment shows that the zero mode, i.e. the field associated with the vanishing eigenvalue, has an effective Hamiltonian that corresponds to a weakly coupled one-component ϕ^4 theory. Thus, the phase transition belongs to the Ising universality class for any N , in agreement with the argument of Ref. [21]. In Ref. [22] it was shown that for $N = \infty$ the transition has mean-field exponents. We reconcile here these two results. If u_t and u_h are the linear thermal and magnetic scaling fields, in the critical limit a generic long-distance quantity \mathcal{O} has a behavior of the form

$$\langle \mathcal{O} \rangle_{\text{sing}} \approx u_t^\sigma f_{\mathcal{O}}(u_t N, u_h N^{3/2}), \quad (164)$$

where $f(x, y)$ is a crossover function. Only if $u_t N \ll 1$ and $u_h N^{3/2} \ll 1$ does one observe Ising behavior. In the opposite limit one observes mean-field criticality. Therefore, the width of the Ising critical region goes to zero as $N \rightarrow \infty$ and, even if the transition is an Ising one for any N , only mean-field behavior is observed for $N = \infty$. The behavior observed at the critical point for $N \rightarrow \infty$ resembles very closely what is observed in medium-range models [26–28, 30], with N playing the role of the interaction range. Our analysis fully confirms the conclusions of Ref. [21].

From a more quantitative point of view, we give explicit expressions for the critical values β_c and p_c and for the nonlinear scaling fields to order $1/N$. Numerical results are given for the Hamiltonian introduced in Ref. [21] and for mixed $O(N)$ - RP^{N-1} models [20].

Finally, note that, even though the model we consider is two-dimensional, the same discussion can be done in generic dimension d . Indeed, for any d the zero mode has a Hamiltonian that corresponds to a weakly coupled ϕ^4 theory. Thus, for $d < 4$, the phase transition is always Ising-like. Some changes, however, should be done in the scaling equations. For generic $d < 4$, Eq. (72) should be replaced by

$$\chi_n = u^{d/(4-d)} t^{-n(d+2)/4} f_n^{\text{symm}}(u t^{-(4-d)/2}, u h^{-2(4-d)/(2+d)}). \quad (165)$$

Also the expressions for the renormalization constants r_c and h_c should be changed: for instance, in three dimensions we also expect contributions from two-loop graphs, as it happens in medium-range models [29]. The main result of this paper, i.e. the fact that the width of the Ising critical region goes to zero as $N \rightarrow \infty$, holds in any $d < 4$. However, in generic dimension d the natural scaling variables are $(p - p_{\text{crit}})N^{2/(4-d)}$ and $(\hat{u}_h - \hat{u}_{h,\text{crit}})N^{3/(4-d)}$. Thus, for $d > 2$ the Ising critical region shrinks faster as N increases: in three dimensions as N^{-2} and N^{-3} in the thermal and magnetic directions respectively.

A Identities among effective vertices

In this Section we wish to derive a general set of identities for the zero-momentum vertices. For this purpose we first rewrite the propagator and the vertices by considering the constants α , γ , and τ appearing in Eq. (6) as independent variables *without* assuming the saddle-point relations. We define the integrals

$$\mathcal{A}_{ij,n}(\alpha, \gamma) = \int_{\mathbf{q}} \frac{\cos^i q_x \cos^j q_y}{(\gamma - \alpha \sum_{\mu} \cos q_{\mu})^n}. \quad (166)$$

It is easy to verify that, if α and γ are replaced by their saddle-point values, $\alpha = 2W'(\bar{\tau})$ and $\gamma = (4 + m_0^2)W'(\bar{\tau})$, then

$$\mathcal{A}_{ij,n}(\alpha, \gamma) = \frac{1}{[W'(\bar{\tau})]^n} \int_{\mathbf{q}} \frac{\cos^i q_x \cos^j q_y}{(\hat{q}^2 + m_0^2)^n}. \quad (167)$$

In terms of α , γ , and τ the propagator is simply obtained by using Eq. (32) and replacing $A_{i,j}(0, m_0^2)/[W'(\bar{\tau})]^2$ with $\mathcal{A}_{ij,2}(\alpha, \gamma)$. Let us now consider a generic n -leg vertex $V_{A_1, \dots, A_n}^{(n)}$ at zero momentum. It is easy to verify that the only nonvanishing terms with $A_i = 4$ or $A_i = 5$ for some i are (in this Section we do not explicitly write the momentum dependence since in all cases we are referring to zero-momentum quantities)

$$V_{4, \dots, 4}^{(n)} = V_{5, \dots, 5}^{(n)} = -\beta W^{(n)}(\tau). \quad (168)$$

If all $A_i \leq 3$, Eq. (36) gives

$$V_{A_1, \dots, A_n}^{(n)} = \frac{1}{2} (-1)^{i_2 + i_3 + n + 1} (n-1)! \mathcal{A}_{i_2 i_3, n}(\alpha, \gamma), \quad (169)$$

where i_2 (resp. i_3) is the number of indices equal to 2 (resp. to 3). These expressions allow us to obtain simple recursion relations for the vertices.

$$\begin{aligned} V_{1, A_1, \dots, A_n}^{(n+1)} &= \frac{\partial}{\partial \gamma} V_{A_1, \dots, A_n}^{(n)}, \\ V_{2, A_1, \dots, A_n}^{(n+1)} + V_{3, A_1, \dots, A_n}^{(n+1)} &= \frac{\partial}{\partial \alpha} V_{A_1, \dots, A_n}^{(n)}, \\ V_{4, 4, \dots, 4}^{(n+1)} &= \frac{\partial}{\partial \tau} V_{4, \dots, 4}^{(n)}, \\ V_{5, 5, \dots, 5}^{(n+1)} &= \frac{\partial}{\partial \tau} V_{5, \dots, 5}^{(n)}, \end{aligned} \quad (170)$$

where α , γ , τ , and β are considered as independent variables. These relations also apply to the case $n = 2$, once we identify $V_{AB}^{(2)} = P_{AB}^{-1}$.

Let us consider the projector on the zero mode v_A , cf. Eq. (45). Keeping into account that the symmetry of the matrix at zero momentum implies $v_2 = v_3$ and $v_4 = v_5$, we obtain

$$\sum_B v_B V_{B,A_1,\dots,A_n}^{(n+1)} = \left(v_1 \frac{\partial}{\partial \gamma} + v_2 \frac{\partial}{\partial \alpha} + v_4 \frac{\partial}{\partial \tau} \right) V_{A_1,\dots,A_n}^{(n)}. \quad (171)$$

Close to the critical point we can write, cf. Eq. (41),

$$v_A = \hat{z}_A + O(s_1) \quad (172)$$

with

$$\hat{z}_A = K \left(2 \frac{\mathcal{A}_{01,2}}{\mathcal{A}_{00,2}}, 1, 1, \frac{1}{2W''(\tau)}, \frac{1}{2W''(\tau)} \right), \quad (173)$$

where K is such to have $\sum_B \hat{z}_B^2 = 1$. Now, let us note that at the saddle point we have (we now think of α , τ , and γ as functions of m_0^2 and p)

$$\begin{aligned} \frac{\partial \gamma}{\partial m_0^2} &= W'' \frac{\partial \tau}{\partial m_0^2} \left(4 + m_0^2 - \frac{B_1}{B_2} \right) + O(s_1) = \frac{2W''}{K} \frac{\partial \tau}{\partial m_0^2} \hat{z}_1 + O(s_1), \\ \frac{\partial \alpha}{\partial m_0^2} &= 2W'' \frac{\partial \tau}{\partial m_0^2} = \frac{2W''}{K} \frac{\partial \tau}{\partial m_0^2} \hat{z}_2 + O(s_1), \end{aligned} \quad (174)$$

where we have used Eqs. (40) and (34). Thus, if we define

$$C \equiv \left(\frac{2W''}{K} \frac{\partial \tau}{\partial m_0^2} \right)^{-1}, \quad (175)$$

we can rewrite

$$\begin{aligned} \sum_B v_B V_{B,A_1,\dots,A_n}^{(n+1)} &= C \left(\frac{\partial \gamma}{\partial m_0^2} \frac{\partial}{\partial \gamma} + \frac{\partial \alpha}{\partial m_0^2} \frac{\partial}{\partial \alpha} + \frac{\partial \tau}{\partial m_0^2} \frac{\partial}{\partial \tau} \right) V_{A_1,\dots,A_n}^{(n)} + O(s_1) \\ &= C \frac{\partial}{\partial m_0^2} V_{A_1,\dots,A_n}^{(n)} + O(s_1), \end{aligned} \quad (176)$$

where in the last term α , γ , τ , and β take their saddle-point values in terms of m_0^2 and p .

To go further we compute the derivative of \hat{z}_A with respect to m_0^2 . Since

$$\sum_B P_{AB}^{-1} \hat{z}_B = O(s_1), \quad (177)$$

we have

$$\sum_B P_{AB}^{-1} \frac{\partial \hat{z}_B}{\partial m_0^2} = - \sum_B \frac{\partial P_{AB}^{-1}}{\partial m_0^2} \hat{z}_B + O(s_2), \quad (178)$$

where $s_2 = \partial^2 \beta / \partial (m_0^2)^2$ also vanishes at the critical point.

Since $\sum_A \hat{z}_A^2 = 1$, $\partial \hat{z}_B / \partial m_0^2$ belongs to the subspace orthogonal to \hat{z}_A . Thus, if P_{AB}^\perp is the inverse of P^{-1} in the massive subspace, we obtain

$$\frac{\partial \hat{z}_A}{\partial m_0^2} = - \sum_{BC} P_{AB}^\perp \frac{\partial P_{BC}^{-1}}{\partial m_0^2} \hat{z}_C + O(s_2) = -\frac{1}{C} \sum_{BCD} P_{AB}^\perp V_{BCD}^{(3)} \hat{z}_C \hat{z}_D + O(s_2). \quad (179)$$

Finally, let us compute

$$\begin{aligned} \hat{V}_{1,\dots,1}^{(n+1)} &= C \sum_{A_1, \dots, A_n} z_{A_1} \dots z_{A_n} \frac{\partial}{\partial m_0^2} V_{A_1, \dots, A_n}^{(n)} + O(s_1) \\ &= C \frac{\partial}{\partial m_0^2} \hat{V}_{1, \dots, 1}^{(n)} - nC \sum_{A_1, \dots, A_n} z_{A_1} \dots z_{A_{n-1}} V_{A_1, \dots, A_n}^{(n)} \frac{\partial z_{A_n}}{\partial m_0^2} + O(s_1) \\ &= C \frac{\partial}{\partial m_0^2} \hat{V}_{1, \dots, 1}^{(n)} + n \sum_{ab} \hat{V}_{11a}^{(3)} \hat{P}_{ab} \hat{V}_{b1\dots 1}^{(n)} + O(s_2), \end{aligned} \quad (180)$$

where the indices a and b run from 1 to 4 over the massive subspace. Since s_2 is of order $p - p_c$ and $m_0^2 - m_{0c}^2$ this relation implies an identity only at the critical point. However, for $n = 2$ we can obtain another relation. Since $V_{AB}^{(2)} = P_{AB}^{-1}$ and $\sum_B P_{AB}^{-1} z_B = O(s_1)$ we do not need Eq. (179) and obtain directly

$$\hat{V}_{111} = C \frac{\partial}{\partial m_0^2} P_{11}^{-1} + O(s_1). \quad (181)$$

The presence of corrections of order s_1 gives rise to two critical-point identities. Since $P_{11}^{-1} \sim s_1$ we obtain

$$\begin{aligned} \hat{V}_{111} &= 0, \\ \frac{\partial}{\partial m_0^2} \hat{V}_{111} &= C \frac{\partial^2}{\partial (m_0^2)^2} P_{11}^{-1}, \end{aligned} \quad (182)$$

where all quantities are computed at the critical point.

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